Math 554

1/22/96

Defn. The *real numbers* are defined to be a set $I\!R$ with two binary operations $(+, \cdot)$ which satisfy the following properties: Given any a, b, c in $I\!R$

1. $a + (b + c) = (a + b) + c$.
2. $a + b = b + a$.
3. $\exists 0 \in I\!R \ni a + 0 = a, \forall a \in I\!R$.
4. for each $a \in I\!R$, $\exists(-a) \in I\!R$, so that $a + (-a) = 0$.
5. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
6. $a \cdot b = b \cdot a$.
7. $\exists 1 \in I\!R \ni 1 \neq 0$ and $a \cdot 1 = a, \forall a \in I\!R$.
8. for each $a \in I\!R$ with $a \neq 0$, $\exists a^{-1} \in I\!R$, so that $a \cdot a^{-1} = 1$.
9. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Moreover, there is a distinguished subset $I\!P$ (the positive cone) of $I\!R$ with the following properties: Given any a, b in $I\!P$,

a. $a + b \in I\!P$,
b. $a \cdot b \in I\!P$,
c. For each a in $I\!R$, exactly one of the following properties holds:
  i) $a \in I\!P$,
  ii) $-a \in I\!P$,
  iii) $a = 0$.

Finally, $I\!R$ must satisfy the **least upper bound** property, that is, each nonempty subset of $I\!R$ which has an *upper bound* has a *least upper bound*. These terms are defined shortly.

Notation:

'$b - a$' is defined as $b + (-a)$.
'$a < b$' means that $b - a \in I\!P$.
'$a \leq b$' means either $a < b$ or $a = b$.
The fraction '$\frac{a}{b}$' means $a \cdot b^{-1}$.

Lemma. For each $a \in I\!R$,

(i.) $(-a) = (-1) \cdot a$.
(ii.) $0 < 1$.
(iii.) if $0 < a$, then $(-a) < 0$.

Proof. Since

$$a + ((-1) \cdot a) = (1 \cdot a) + ((-1) \cdot a) = a \cdot (1 + (-1)) = 0 \cdot a = 0,$$

it follows from Homework Problem 2.3 that $\alpha := (-1) \cdot a$ is an additive inverse for a. But additive inverses are unique (from your Homework Problem 2.2), so the conclusion of part (i) follows.

To prove part (ii), we assume to the contrary, i.e. that $1 \notin I\!P$. By the definition, $1 \neq 0$, so $(-1) \in I\!P$ and therefore $0 < -1$. But $(-1) \cdot (-1) = -(-1) = 1$ by part (i) and the HW Problem that additive inverses are unique. This shows that $1 \in I\!P$ by property (b) of the positive cone and the assumption that $(-1) \in I\!P$. Contradiction, by the tricotomy property (c).

For part (iii), observe that $0 < a$ means $a \in \mathbb{P}$. Since additive inverses are unique, then $-(-a) = a$, and so $(0 - (-a)) = -(-a) \in \mathbb{P}$. This is equivalent to the statement $(-a) < 0$. $\square$

Homework #2 (Due Friday, Jan. 26)

1. Show that the additive (or multiplicative) identity is unique.
2. Show that additive (or multiplicative) inverses are unique.
3. Prove that $a \cdot 0 = 0$ for each a in $\mathbb{R}$.
4. Prove that $a < b$ and $0 < c$ implies that $a \cdot c < b \cdot c$.
5. For each pair of real numbers a, b, prove that exactly one of the following properties hold:

   i) $a < b$,
   ii) $b < a$,
   iii) $a = b$.

6. Prove that if $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.
7. Prove that the interval (a, b) is uncountable.
8. Prove that $-(a + c) = (-a) + (-c)$.

Defn. A real number α is said to be an *upper bound* for $A \subseteq \mathbb{R}$ if

$$a \leq \alpha, \quad \forall a \in A.$$

A real number M is said to be a *least upper bound* for $A \subseteq \mathbb{R}$ if

1. M is an upper bound for A
2. if α is any upper bound for A, then $M \leq \alpha$.

In this case, the *supremum of* A ($=: \sup A$) is defined as M. The definitions are similar for *lower bound*, *greatest lower bound* and $\inf A$, respectively.

Lemma. Suppose that A is a nonempty subset of $\mathbb{R}$, with least upper bound M, then for every $\epsilon > 0$, there exists $a \in A$ such that

$$M - \epsilon < a \leq M.$$

Proof. Since $0 < \epsilon$, then $M - \epsilon < M$. This shows that $M - \epsilon$ cannot be an upper bound for A. Hence there is a member of A, call it a, so that $M - \epsilon < a$. $\square$

Theorem. (Archimedean Property) Suppose a, b are positive real numbers, then there exists $n \in \mathbb{N}$ such that $b < n \cdot a$.
Proof. Suppose to the contrary that $na < b$ for all $n \in \mathbb{N}$, then it follows that $\alpha := b/a$ is an upper bound for the natural numbers. Let M be the least upper bound. By the lemma, $1/2 > 0$, so there exists a natural number N so that $M - 1/2 < N$. But then, $M < N + 1/2 < N + 1$, which shows that M is not an upper bound for $\mathbb{N}$. Contradiction. $\square$

Corollary. The natural numbers are not bounded.

Corollary. Given any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$.

Homework #3 (Due Monday, Jan. 29)

1. Prove that least upper bounds are unique.
2. Prove the first corollary to the Archimedean Property.
3. Prove the *greatest lower bound property* for the real numbers, i.e., each nonempty subset of $\mathbb{R}$ which has a lower bound, has a greatest lower bound.

Notation: Next we define intervals of real numbers.

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$ is called the *open interval* with endpoints a, b.
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$ is called the *closed interval* with endpoints a, b.
$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$ and $[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$ are called the *half open intervals* with endpoints a, b.

Theorem. Suppose that I is an interval with endpoints a, b and $a < b$, then I contains a rational number.
Proof. Define the *length of* I by $\ell := b - a$. By the previous corollary, there exists $n_o \in \mathbb{N}$ such that $0 < 1/n_o < \ell$. Let $A := \{k |\ k \text{ an integer and } k/n_o < a\}$. A is nonempty, since the negative integers are not bounded from below. Let k_o belong to A. Set $B := \{k |\ k \text{ an integer and } k \geq k_o\} \cap A$. Also, A is bounded from above by $a \cdot n_o$, which shows that B is in fact a finite set of integers. Let K be the largest member of B and therefore of A, then $K + 1 \notin A$. Let $r := (K + 1)/n_o$, then

$$a < \frac{K+1}{n_o} < \frac{K}{n_o} + \ell \leq a + (b - a) = b,$$

which shows that the rational $r \in (a, b) \subseteq I$. $\square$

Corollary. Each interval with nonzero length contains an infinite number of rationals.

Defn. A real number is said to be *irrational* if it is not rational.

Corollary. Each interval with nonzero length contains an uncountably infinite number of irrationals.

We establish a few other facts about irrational numbers and also prove directly that each interval of positive length contains an infinite number of irrationals.

Lemma. The product of a nonzero rational with an irrational is irrational.
Proof. Suppose that $q_1 \cdot \alpha = q_2$, where q_1, q_2 are rational and α is irrational. Since $q_1 \neq 0$, then $\alpha = q_2/q_1$ and it follows that α is rational. Contradiction. $\square$

Lemma. If m is an odd integer, then m^2 is odd.
Proof. If m is odd, then there exists an integer k such that $m = 2k + 1$. In this case $m^2 = 2(2k^2 + 2k) + 1$. $\square$

Lemma. $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2} = m/n$ where m, n are integers with $n > 0$. We may assume that the rational is in lowest terms (i.e. m and n have no common factors). Squaring the equation and multiplying by n^2, we obtain that $m^2 = 2n^2$. This shows that m^2 is even. By the lemma m must be even and equivalently that it contains 2 as a factor. This shows $4k^2 = 2n^2$ for some integer k. Consequently, n is even and 2 appears as one of its factors. Contradiction, since m/n was supposed to be in lowest terms. $\square$
Proof. (Another proof of the 'density' of the irrationals) Let a, b be the endpoints of the interval I. Consider the interval $(a/\sqrt{2}, b/\sqrt{2})$. It has length $(b-a)/\sqrt{2} > 0$, and so contains a nonzero rational number q. It follows that $q\sqrt{2}$ is between a and b and hence belongs to I. $\square$

Defn. The absolute value of a real number a is defined by

$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Lemma. The *absolute value* function has the following properties:

1. $|a| \geq 0$ for all $a \in \mathbb{R}$.
2. $\pm a \leq |a|$
3. $|-a| = |a|$
4. $|a \cdot b| = |a| \cdot |b|$
5. $|a + b| \leq |a| + |b|$
6. $||b| - |a|| \leq |b - a|$

Proof. To prove property 1, 'case it out.' Either $a \geq 0$ or $a < 0$. In the first case $|a| = a \geq 0$. In the second case, $|a| = -a > 0$, since $a < 0$. Property 2 follows similarly. Properties 3 and 4 are left for the student.

To prove property 5, recall that either $c := a + b$ satisfies $c \geq 0$ or $c < 0$. In the first case, $|c| = c = a + b \leq |a| + |b|$. In the second case, $|c| = -c = -a + (-b) \leq |a| + |b|$. This establishes property 5.

To prove property 6, use the fact that $|b| = |(b-a) + a| \leq |b - a| + |a|$ and subtract $|a|$ from each side. This shows that $|b| - |a| \leq |b - a|$. By symmetry in a and b, one proves that $|a| - |b| \leq |a - b| = |b - a|$. $\square$

Homework #4 (Due Wednesday, Jan. 31)

1. Work problem 1.46 in the text.
2. Work problem 1.54 in the text.

Graduate Student's Additional Homework

1. Prove $0 < a < b$ that implies that $a^2 < b^2$.
2. Prove that the number $\alpha := \sup\{x \in \mathbb{R} | x^2 < 2\}$ is the unique positive number that satisfies the equation $\alpha^2 = 2$.
3. Give a definition for square roots of nonnegative real numbers.
4. Prove that $|a| = \sqrt{a^2}$ for all $a \in \mathbb{R}$.
