MATH 554/703I - ANALYSIS I
TEST 1 – SEPTEMBER 20, 2001

1	(16 *pts*)
2	(10 *pts*)
3	(15 *pts*)
4	(24 *pts*)
5	(10 *pts*)
6	(15 *pts*)
7	(10 *pts*)

Name: Key

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1. Pick exactly two of the following three parts to work: Suppose that F is an ordered field,

   (a) prove for each $a \in F$, $a \cdot 0 = 0$.

   Since 0 is the additive identity for F, then $0 = 0 + 0$. Hence $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$ follows by the distributive property of the field. Adding the additive inverse of $a \cdot 0$ to this identity gives

   $$0 = 0 + (a \cdot 0)$$

   which implies $0 = a \cdot 0$. ∎

   (b) prove that $0 < 1$.

   By the trichotomy property either $0 < 1$ (then done), $0 = 1$ ($\divideontimes$ to definition of F), or $1 < 0$. If the last case were to hold, i.e. $1 < 0$, then $0 - 1 \in \mathbb{P}$. Multiplying two positive numbers gives

   $$(-1) \cdot (-1) \in \mathbb{P}.$$

   But $(-1)(-1) = -(-1) = 1$, so $1 \in \mathbb{P}$ $\divideontimes$ $-1 \in \mathbb{P}$. ∎

   (c) if $a < b$ and $c < d$, prove that $a + c < b + d$.

   Since $a < b$, then $(b-a) \in \mathbb{P}$. Similarly $c < d \Rightarrow (d-c) \in \mathbb{P}$. The sum of two elements in $\mathbb{P}$ belongs to $\mathbb{P}$, so

   $$(b-a) + (d-c) \in \mathbb{P}.$$

   But $(b-a) + (d-c) = (b+d) - (a+c)$ by the field properties, so $(b+d) - (a+c) \in \mathbb{P}$, which implies $a + c < b + d$. ∎

2. (a) Give a precise definition for a set to be finite.

A set A is finite means either $A = \phi$ or $\exists\ N \in \mathbb{N} \ni$ $\{1, 2, \ldots, N\}$ is in 1:1 correspondence with A. That is $\exists$ a function f such that

$$f : \{1, 2, \ldots, N\} \xrightarrow[\text{onto}]{1:1} A.$$

(b) Give a precise definition for a set to be countably infinite.

A set A is countably infinite means $\exists$ a function f

$$f : \mathbb{N} \xrightarrow[1:1]{\text{onto}} A.$$

3. Let A be an nonempty subset of $\mathbb{R}$.

a.) Define 'upper bound' for A.

$M \in \mathbb{R}$ is called an upper bound for A if

$$a \leq M, \quad \forall a \in A.$$

b.) Define 'least upper bound' for A.

$\gamma \in \mathbb{R}$ is called a least upper bound for A if

1) γ is an upper bound for A

2) if M is any upper bound for A, then $\gamma \leq M$.

c.) Prove that least upper bounds are unique.

Suppose γ_1 & γ_2 are both least upper bounds for A, then γ_2 is an upper bound for A. γ_1 a least upper bound for $A \Rightarrow \boxed{\gamma_1 \leq \gamma_2}$ Reverse the roles of γ_1 & γ_2 (i.e. γ_1 an upper bound for A & γ_2 a lub A) to give that $\boxed{\gamma_2 \leq \gamma_1}$. Hence $\gamma_1 = \gamma_2$ ∎

4. a. State and prove the Archimedean principle.

Archimedian Principle If a, b are any two positive real numbers then $\exists\ n \in \mathbb{N} \ni b < na$.

proof We proceed using a contrapositive argument. Suppose not, that is in fact $b \geq na$, $\forall n \in \mathbb{N}$. Hence $M := b/a$ is an upper bound for $\mathbb{N}$. Let γ be the least upper bound by the Completeness Axiom. Now $\gamma - \frac{1}{2} < \gamma$ so $\gamma - \frac{1}{2}$ cannot be an upper bound for $\mathbb{N}$. Hence $\exists\ n \in \mathbb{N} \ni \gamma - \frac{1}{2} < n$. But then $\gamma < n + \frac{1}{2} < n+1 \in \mathbb{N}$. ※ γ must be an upper bound for $\mathbb{N}$. ∎

b. Prove that for each $\epsilon > 0$, there exists a natural number N such that for all $N \leq n$ there holds $0 < \frac{1}{n} < \epsilon$. By the Archimedean principle $\exists\ N \in \mathbb{N} \ni 1 < N \cdot \varepsilon$, by choosing $a = \varepsilon$ and $b = 1$. It follows that $\boxed{1/N < \varepsilon}$. If $n \in \mathbb{N}$ & $n \geq N$ then $0 < \frac{1}{n} < \frac{1}{N}$.

This then implies that

$$0 < \frac{1}{n} < \frac{1}{N} < \varepsilon \quad , \text{if } n \geq N. \ ∎$$

5. Negate the statement:

for each $\epsilon > 0$ there is a natural number N such that for every $n \geq N$ it is implied that $|a_n - a| < \epsilon$

$$\sim\left(\forall\, \varepsilon > 0 \left(\exists\, N \in \mathbb{N} \ni \left(\forall\, n \geq N \Rightarrow |a_n - a| < \varepsilon\right)\right)\right)$$

$$\Rightarrow \exists\, \varepsilon_0 > 0 \ni \sim\left(\exists\, N \in \mathbb{N} \ni \left(\forall\, n \geq N \Rightarrow |a_n - a| < \varepsilon_0\right)\right)$$

$$\Rightarrow \boxed{\exists\, \varepsilon_0 > 0 \ni \forall\, N \in \mathbb{N} \left(\exists\, n \geq N \ni |a_n - a| \geq \varepsilon_0\right).}$$

6. For $a > 0$, and all natural numbers n, prove that

$$(p_n) \qquad 1 + na \leq (1 + a)^n$$

<u>Proof</u> We use induction on n.

<u>Case $n=1$</u> $\quad 1 + 1a = (1+a)^1$ since both sides equal $1+a$.

<u>Induction step</u> Suppose (p_n) is true for n. We need to prove that (p_{n+1}) is true. But (p_n) is true & $(1+a) > 0$ so

$$(1+a)^{n+1} = (1+a)^n (1+a) \geq (1+na)(1+a)$$

$$= 1 + na + a + na^2 > 1 + (n+1)a$$

since $na^2 > 0$. Hence (p_{n+1}) is true & the induction step is complete.

$\therefore$ (p_n) is true $\forall n \in \mathbb{N}$. $\blacksquare$

7. Prove that if $0 < r < 1$ and $\epsilon > 0$, then there exists a natural number n so that $r^n < \epsilon$. (Hint: Problem #6) Let $a = \frac{1}{r} - 1$ in Problem #6. Since $0 < r < 1$, then $1 < \frac{1}{r}$ & so $a > 0$. Notice $1 + a = \frac{1}{r}$ so the previous problem implies

$$1 + na \leq \left(\frac{1}{r}\right)^n, \quad \forall n \in \mathbb{N}.$$

Hence

$$0 < r^n < \frac{1}{a} \frac{1}{n}$$

or

(*) $$0 < r^n < \left(\frac{r}{1-r}\right) \frac{1}{n} \quad , \forall n \in \mathbb{N}.$$

By the Archimedean Principle with the numbers $\frac{r}{1-r}$ & $\varepsilon > 0$ we may find $n \in \mathbb{N} \ni$

$$\frac{r}{1-r} < n\varepsilon$$

Therefore by this and inequality (*)

$$0 < r^n < \frac{r}{1-r} \frac{1}{n} < \varepsilon. \quad \blacksquare$$