Math 554 HW #6 Solutions Oct. 18, 2001

1. Prove $\mathbb{R}$ & ϕ are closed sets.

pf Both $\mathbb{R}$ & ϕ are open sets, so their complements ϕ & $\mathbb{R}$ are closed sets: $\mathbb{R}$ is open because $\forall x \in \mathbb{R}$ $N_1(x) \subseteq \mathbb{R}$.

ϕ is open because $\forall x \in \phi$ (there are none!), $N_1(x) \subseteq \phi$. $\blacksquare$

2. Let $\{C_\alpha\}_{\alpha \in A}$ be a collection of closed sets. Then $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is a collection of open sets, where $\mathcal{O}_\alpha := C_\alpha^c$ (i.e. complement of C_α). By de Morgan's law

$$\bigcap_{\alpha \in A} C_\alpha = \bigcap_{\alpha \in A} \mathcal{O}_\alpha^c = \Big(\bigcup_{\alpha \in A} \mathcal{O}_\alpha\Big)^c.$$

But we know $\bigcup_{\alpha \in A} \mathcal{O}_\alpha$ is open (arbitrary unions of open sets are open) & $\bigcap_{\alpha \in A} C_\alpha$ is the complement of this open set, so must be closed. $\blacksquare$

3. Similar reasoning to #2: $\bigcup_{i=1}^n C_i = \bigcup_{i=1}^n \mathcal{O}_i^c = \left(\bigcap_1^n \mathcal{O}_i\right)^c$ where $\mathcal{O}_i := C_i^c$, $1 \le i \le n$.

4. If $C = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$, then $C^c = (-\infty, 0) \cup \left(\bigcup_{i=1}^\infty (\frac{1}{i+1}, \frac{1}{i})\right) \cup (1, \infty)$ is open, so C must be closed.

5. 0 is the only limit point of the set $A := \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$: 0 is a limit point of A, since $0 < |\frac{1}{n} - 0| < \varepsilon$ if $n \ge N$, i.e.

$$A \cap N_\varepsilon(0) \setminus \{0\} \ne \phi \quad \forall \varepsilon > 0.$$

& all but a finite # of elements of A belong to every neighborhood of $\{0\}$.

6. Limit points of rationals in $(0,1]$ consists of all points in $[0,1]$. We already know $\forall$ irrational $x \in (0,1]$ an infinite number of rationals belong to each nbhd of x. For every rational x consider rationals of the form $x + \frac{1}{n}$ ... Just make sure the rationals you choose belong to $(0,1]$.

7. Ans. $[-1, 2] \cup [5, 6]$.