Homework # 4
Math 554 / 703 I

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#1 a) $A = \mathbb{N}$ & B = odd positive integers.

If we define $f(a) = 2a - 1$, $a \in \mathbb{N}$

then ⓐ $f(a)$ is odd & pos. $\forall a \in \mathbb{N} \Rightarrow f: \mathbb{N} \to B$

ⓑ f is 1:1 since $2a_1 - 1 = 2a_2 - 1 \Rightarrow a_1 = a_2$

& ⓒ f is onto since $a = \frac{b+1}{2} \in \mathbb{N}$ if $b \in B$.

d.) $A = (1,2)$ & $B = (1,6)$

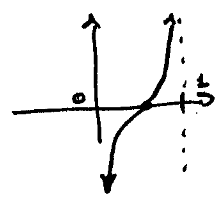
Let $f(x) = 5x - 4$, then f is 1:1 & onto B with domain A.

ⓐ $1 < x < 2 \Rightarrow 1 < 5x - 4 < 6$ shows $f: A \to B$

ⓑ if $5x_1 - 4 = 5x_2 - 4$ then $x_1 = x_2$ shows f is 1:1.

ⓒ if $1 < y < 6$ then $x = \frac{y+4}{5} \in (1,2)$ & $f(x) = y$ which shows f is onto.

f.) $A = \mathbb{R}$ & $B = (0,1)$



Soln 1 Let $f(x) = \frac{x - \frac{1}{2}}{x(1-x)}$, $0 < x < 1$.

$f: B \xrightarrow[\text{onto}]{1:1} A$ ⓐ $f(x) \in \mathbb{R}$ if $x \neq 0, 1$.

ⓑ $f \uparrow$ (strictly increasing) $\Rightarrow f$ is 1:1. ; ⓒ f onto: If $y \in \mathbb{R}$ & we set $\frac{x - \frac{1}{2}}{x(1-x)} = y$, then we can solve for x in terms of y by the quadratic formula.

Soln 2 Note: $\cot(\pi x) : (0,1) \xrightarrow[\text{onto}]{1:1} \mathbb{R}$

#9 Suppose A is countable & B is uncountable, then prove B is uncountable.

proof Suppose not, i.e. $C = A \cup B$ is countable. But any subset of C must be countable. In particular $B \subseteq C$ so B is countable ※ ☒

Homework #4 (continued)

#1. If $1<a$ & $1<b$, then $1<ab$.

proof $1<a \Rightarrow (a-1)>0$. Similarly $(b-1)>0$. Hence $0<(a-1)(b-1) = ab-a-b+1$, and so

$$1<a<a+(b-1)<ab \quad ▨$$

#2 If $0<a$ & n is a natural number, then $1<(1+a)^n$.

proof Proof by induction. If $n=1$, then statement is true since $1<1+a=(1+a)^1$. Suppose then that the statement is true for n, i.e. $1<(1+a)^n$. We show that it is true for $n+1$. Indeed $(1+a)>1$ & $(1+a)^n>1$ so by #1

$$(1+a)^{n+1}>1. \quad ▨$$

#3 If $a>0$ & $n\in\mathbb{N}$, then Bernoulli's inequality

$$(p_n) \qquad 1+na \leq (1+a)^n$$

holds.

proof If $n=1$, then trivial. Suppose (p_n) is true for n, then

$$(1+a)^{n+1} = (1+a)^n(1+a) \geq (1+na)(1+a)$$
$$= 1+(n+1)a+na^2 \geq 1+(n+1)a.$$

Hence (p_{n+1}) holds & the induction step is verified. ▨