Math 554 | HW # 3 Solutions | Sept 13, 2001

Problem If $a \neq 0$ is rational & b is irrational, then ab is irrational.

proof Suppose not, i.e. Suppose $c = ab$ is rational, then

$$b = ca^{-1} \in \mathbb{Q} \quad \times$$

since $\mathbb{Q}$ is a field. But b is irrational. ∎

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#8 Suppose A is bounded from above & is nonempty. If γ_1 & γ_2 are both least upper bounds of A, then

$$\gamma_1 \le \gamma_2$$

since γ_1 is a l.u.b. of A & γ_2 is an upper bound of A.

Similarly, γ_2 is a l.u.b. of A & γ_1 is an upper bound of A implies

$$\gamma_2 \le \gamma_1 .$$

Hence $\gamma_1 = \gamma_2$. ∎

#10 If $x \in (a,b)$, then $a < x < b$ so a is a lower bound for (a,b) & b is an upper bound. Let γ be the least upper bound, then $\gamma \le b$. If $\gamma < b$, then $x = \frac{\gamma + b}{2}$ satisfies

$$a < \gamma < x < b .$$

But then $x \in (a,b)$ & $\gamma < x$ so γ is not an upper bound for (a,b). $\times$ ∎

Similarly $a = \inf(a,b)$.

#20 Let $x \in \mathbb{R}$. Prove: $\forall \varepsilon > 0 \ \exists r \in \mathbb{Q} \ni 0 < |x - r| < \varepsilon$.

proof Fix x & ε as above. Consider the interval $(x, x+\varepsilon)$. $\exists r \in \mathbb{Q} \ni x < r < x + \varepsilon$. This rational number r satisfies

$$0 < r - x = |x - r| = r - x < \varepsilon . \quad ∎$$