Math 554 - Analysis I
Sample Final Exam – Dec. 5, 2001

Name ______________________

Directions: You must show your work for partial credit.

1. a.) State the Archimedean Principle.
   b.) Prove that for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$.

2. a.) Define the least upper bound of a set.
   b.) Prove that γ is the least upper bound of a nonempty set A if for each $\epsilon > 0$, there exists $a \in A$ such that
   $$\gamma - \epsilon < a \leq \gamma.$$

3. a.) Give the $\epsilon - \delta$ definition of continuity of f at x_0.
   b.) Prove that a function f is continuous if and only if $f^{-1}[\mathcal{O}]$ is relatively open for each open set $\mathcal{O}$.
   c.) Prove that the composition of two continuous functions is continuous. (You may use any properties of continuous functions, but be certain to explain what you are doing.)

4. Show that the continuous image of a closed interval $[a, b]$ is a closed interval, i.e. if $f : [a, b] \to R$ is continuous, then there exists a closed interval $[c, d]$ such that $f([a, b]) = [c, d]$.

5. Proved that a compact set is bounded.

6. State and sketch the proof of the Heine-Borel Theorem.

7. Suppose that f is continuous on a compact set K, then prove that f is uniformly continuous.

Work 2 of the following 3 problems:

8. State and sketch the proof of the Mean (or Intermediate) Value Theorem for integrals.

9. a.) Define differentiability of a function f at a point a.
   b.) Show that a function is differentiable at a if and only if there exists a function $\eta(x)$ such that

$$f(x) = f(a) + (x-a)f'(a) + (x-a)\eta(x)$$

   and

$$\lim_{x \to a} \eta(x) = 0.$$

   c.) Prove that differentiability of f at a implies that f is continuous at a.

10. a.) State both parts of the Fundamental Theorem of Calculus.
    b.) Prove one of the two parts.