MATH 554 – RIEMANN INTEGRATION

Handout #9b (Dec. 4)

Corollary. If f is Riemann integrable on $[a, b]$, then so is $-f$ and

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx.$$

Proof. We use the condition (*) and note that

$$U(-f, P) = -L(f, P), \quad L(-f, P) = -U(f, P)$$

and so

$$U(-f, P) - L(-f, P) = U(f, P) - L(f, P) < \epsilon.$$

Since this holds for each positive ϵ, it follows that $\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$. □

Note. This result shows that for a function to be Riemann integrable it is enough to find, for each positive ϵ, a partition fine enough that the corresponding upper and lower sums are within ϵ units of one another. In this case the Riemann integral is within ϵ units of either approximating sum.

Examples: You should go through the following two examples on your own to make sure you understand the mechanics.

1. $\int_a^b k\,dx = k(b-a)$.
2. $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$. [Hint: Use the result proved earlier: $\sum_{i=1}^n i = n(n+1)/2$.]

Theorem. If f is continuous on $[a, b]$, then f is Riemann-integrable on $[a, b]$.

Proof. We use the condition (*) to prove that f is Riemann-integrable. If $\epsilon > 0$, we set $\epsilon_0 := \epsilon/(b-a)$. Since f is continuous on the compact set $[a, b]$, f is uniformly continuous. Hence there is a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon_0$ if $|y - x| < \delta$. Suppose that $\|P\| < \delta$, then it follows that $|M_i - m_i| \le \epsilon_0$ $(1 \le i \le n)$. Hence

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)\Delta x_i \le \epsilon_0(b-a) = \epsilon. \quad \square$$

The following indented material (definition and resulting corollary) are included for completeness and are not required for the further development. You will not be responsible for these two on the Final.

> **Defn.** A *Riemann sum* for f for a partition P of an interval $[a, b]$ is defined by
>
> $$R(f, P, \boldsymbol{\xi}) := \sum_{j=1}^n f(\xi_j)\Delta x_j$$
>
> where the ξ_j, satisfying $x_{j-1} \le \xi_j \le x_j$ $(1 \le j \le n)$, are arbitrary.

Corollary. Suppose that f is Riemann integrable on $[a, b]$, then there is a unique number γ ($= \int_a^b f(x)dx$) such that for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that if $P \prec P_1, P_2$, then

i.) $\quad 0 \leq U(f, P_1) - \gamma < \epsilon$

ii.) $\quad 0 \leq \gamma - L(f, P_2) < \epsilon$

iii.) $\quad |\gamma - R(f, P_1, \boldsymbol{\xi})| < \epsilon$

where $R(f, P_1, \boldsymbol{\xi})$ is any Riemann sum of f for the partition P_1. In this sense, we can interpret

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} R(f, P, \boldsymbol{\xi}).$$

although we would actually need to show a little more to be precise. *Proof.* Since $L(f, P_2) \leq \gamma \leq U(f, P_1)$ for all partitions, we see that parts i.) and ii.) follow from the definition of the Riemann integral. To see part iii.), we observe that $m_j \leq f(\xi_j) \leq M_j$ and hence that

$$L(f, P_1) \leq R(f, P_1, \boldsymbol{\xi}) \leq U(f, P_1).$$

But we also know that both

$$L(f, P_1) \leq \gamma \leq U(f, P_1)$$

and condition (*) hold, from which part iii.) follows. $\square$

Note. The following theorem is also included for completeness, but will not be needed for our development.

Theorem. If f is monotone on $[a, b]$, then f is Riemann-integrable on $[a, b]$.

Proof. If f is constant, then we are done. We prove the case for f monotone increasing. The case for monotone decreasing is similar. We again use the condition (*) to prove that f is Riemann-integrable. If $\epsilon > 0$, we set $\delta := \epsilon/(f(b) - f(a))$ and consider any partition P with $\|P\| < \delta$. Since f is monotone increasing on $[a, b]$, then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Hence

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta x_i \\ &\leq \|P\| \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &< \delta\,(f(b) - f(a)) = \epsilon. \qquad \square \end{aligned}$$

Theorem. (Monotone Property of the Riemann Integral) Suppose that f and g are Riemann integrable and k is a real number, then

i.) $g \le f$ implies $\int_a^b g\,dx \le \int_a^b f\,dx$.

ii.) $|\int_a^b f\,dx| \le \int_a^b |f|\,dx$

Proof. Property i.) follows directly from the definition of the upper and lower integrals using the inequalities $\sup_I g(x) \le \sup_I f(x)$ and $\inf_I g(x) \le \inf_I f(x)$ for each subinterval I.

Property ii.) is proved by applying property i.) to the inequality

$$-|f| \le f \le |f|,$$

to obtain $-\int_a^b |f|\,dx \le \int_a^b f\,dx \le \int_a^b |f|\,dx$. But this inequality is equivalent to property ii.). □

Defn. We extend the definition of the integral to include general limits of integration. These are consistent with our earlier definition.

1. $\int_a^a f(x)\,dx = 0$.
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Theorem. If f is Riemann integrable on $[a,b]$, then it is Riemann integrable on each subinterval $[c,d] \subseteq [a,b]$. Moreover, if $c \in [a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{3}$$

Proof. We show first that condition (*) holds for the interval $[c,d]$. Suppose $\epsilon > 0$, then by (*) applied to f over the interval $[a,b]$, we have that there exists a partition P of $[a,b]$ such that condition (*) holds. Let $\tilde{P}$ be the refinement obtained from P which contains the points c and d. Let P^* be the partition obtained by restricting the partition $\tilde{P}$ to the interval $[c,d]$, then

$$U(f,P^*) - L(f,P^*) \le U(f,\tilde{P}) - L(f,\tilde{P}) \le U(f,P) - L(f,P) < \epsilon$$

and so f is Riemann integrable over $[c,d]$.

To prove the identity (3), we use the fact that condition (*) holds when f is Riemann integrable. Let $\epsilon > 0$, then for $\epsilon/3 > 0$, we may apply (*) to each of the intervals $I = [a,b], [a,c]$ and $[c,b]$, respectively, to obtain partitions P_I which satisfy

$$0 \le U_I(f,P_I) - \int_I f\,dx \le U_I(f,P_I) - L_I(f,P_I) < \epsilon/3. \tag{4}$$

We let P be the partition of $[a,b]$ formed by the union of the two partitions $P_{[a,c]}, P_{[c,b]}$, and $\tilde{P}$ be the common refinement of P and $P_{[a,b]}$. Observing that

$$U_{[a,b]}(f,\tilde{P}) = U_{[a,c]}(f,\tilde{P}_1) + U_{[c,b]}(f,\tilde{P}_2), \tag{5}$$

we can combine with inequality (4) to obtain

$$\begin{aligned} \left|\int_a^c f\,dx + \int_c^b f\,dx - \int_a^b f\,dx\right| &\le \left|U_{[a,c]}(f,\tilde{P}) - \int_a^c f\,dx\right| + \left|U_{[c,b]}(f,\tilde{P}) - \int_c^b f\,dx\right| \\ &\qquad + \left|U_{[a,b]}(f,\tilde{P}) - \int_a^b f\,dx\right| \\ &< 3\epsilon_0 = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, then equality (3) must hold. $\square$

Corollary. Each bounded, piece-wise continuous function with left and right hand limits at each point of an interval $[a, b]$ is Riemann integrable. Moreover, its integral is the sum of the integrals of the "pieces".

Theorem. (Intermediate Value Theorem for Integrals) If f is continuous on $[a, b]$, then there exists ξ between a and b such that

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Proof. Since f is continuous on $[a, b]$ and for $\eta := \dfrac{\int_a^b f\,dx}{b-a}$ there holds

$$\min_{[a,b]} f(x) \le \eta \le \max_{[a,b]} f(x),$$

then by the Intermediate Value Theorem for continuous functions, there exists a $\xi \in [a, b]$ such that $f(\xi) = \eta$. $\square$

Theorem. (Fundamental Theorem of Calculus, I. Derivative of an Integral) Suppose that f is continuous on $[a, b]$ and set $F(x) := \int_a^x f(y)\,dy$, then F is differentiable and $F'(x) = f(x)$ for $a < x < b$.

Proof. Notice that

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{\int_{x_0}^{x_0+h} f\,dx}{h} = f(\xi)$$

for some ξ between x_0 and x_0+h. Hence, as $h \to 0$, then $\xi = \xi_h$ converges to x_0 and so the displayed difference quotient has a limit of $f(x_0)$ as $h \to 0$. $\square$

Theorem. (Fundamental Theorem of Calculus, Part II. Integral of a Derivative) Suppose that F is function with a continuous derivative on $[a, b]$, then

$$\int_a^b F'(y)\,dy = F(x)|_{x=a}^{x=b} := F(b) - F(a)$$

.

Proof. Define $G(x) := \int_a^x F'(y)\,dy$, and set $H := F - G$. Since the derivative of H is identically zero (Part I of the Fundamental Theorem of Calculus), then the Mean Value Theorem implies that $H(b) - H(a) = 0(b-a) = 0$. Expressing this in terms of F and G gives

$$F(b) - F(a) = G(b) - G(a) = \int_a^b F'(y)\,dy,$$

which establishes the theorem. $\square$

Defn. For a function f, we call any function F, whose derivative is f, an *antiderivative of f*.

Theorem. (Linearity Property of the Riemann Integral) Suppose that f and g are Riemann integrable and k is a real number, then

i.) $\int_a^b k\,f(x)\,dx = k\,\int_a^b f(x)\,dx$

ii.) $\int_a^b f + g\, dx = \int_a^b f\, dx + \int_a^b g\, dx$

Proof. It is a good exercise to prove these directly from the definition and to use the condition (*). The serious student should go through this in detail just for additional practice . We present simpler proofs using the Fundamental Theorems of Calculus. For part ii.) we let F be an antiderivative of f and G be an antiderivative of g, then $H := F + G$ is an antiderivative of $f + g$. Therefore

$$\begin{aligned}\int_a^b f + g\, dx = H(b) - H(a) &= F(b) - F(a) + G(b) - G(a)\\ &= \int_a^b f\, dx + \int_a^b g\, dx.\end{aligned}$$

Part i.) is proved similarly using the corresponding property of differentiation. $\square$