Math 554- 703 I - Analysis I
Existence of Square Roots

Theorem. If a is a nonnegative real number, then there exists a unique positive real number α such that $\alpha^2 = a$. We use the notation $\sqrt{a} := \alpha$.

Lemma. Positive square roots are unique.
Proof. Suppose not. If $x < y$ and x, y are both positive square roots of $a > 0$, then $x^2 < xy < y^2$. But $x^2 = a = y^2$. Contradiction. $\diamond$

Proof of the Theorem. First notice that we may assume without loss of generality that $0 < a < 1$. If $a = 1$, then $\alpha = 1$ is the unique square root of a. If $1 < a$, then $b := 1/a$ is less than 1, and we denote its square root by β. We set $\alpha := 1/\beta$, then $\alpha^2 = 1/\beta^2 = 1/b = a$. Also notice that in the case $0 < a < 1$, the Lemma and its proof imply that $0 < \alpha < 1$.

For $0 < a < 1$, we define the set

$$A := \left\{x > 0 | x^2 \leq a\right\}.$$

Notice that A is nonempty ($a \in A$) and bounded from above by 1, so let $\alpha := \operatorname{lub} A$. Suppose that $\alpha^2 \neq a$.
<u>*Case 1.*</u> If $a < \alpha^2$, then we observe that $\epsilon := \frac{\alpha^2 - a}{2}$ is positive. We claim that $\beta = \alpha - \epsilon$ is an upper bound for A, which would contract the statement that α is the least upper bound. By the definition of ϵ we see that

$$\beta^2 > \alpha^2 - 2\epsilon\alpha > \alpha^2 - 2\epsilon = a \geq x^2$$

for each $x \in A$. Hence β is greater than all $x \in A$.
<u>*Case 2.*</u> If $\alpha^2 < a$, then set $x = \alpha + \epsilon$ where

$$\epsilon = \min\left\{\frac{a - \alpha^2}{2\alpha + 1}, 1\right\}.$$

We claim that $x \in A$ which would contradict that α is the least upper bound of A. Indeed, using the definition of ϵ, we see that

$$x^2 = \alpha^2 + 2\epsilon\alpha + \epsilon^2 \leq \alpha^2 + (2\alpha + 1)\epsilon \leq a. \quad \diamond$$