MATH 554 - ANALYSIS I
TEST 2 – OCTOBER 28, 2008

1	(15 *pts*)
2	(20 *pts*)
3	(5 *pts*)
4	(10 *pts*)
5	(10 *pts*)
6	(20 *pts*)
7	(15 *pts*)
8	(10 *pts*)

Name: ____________________________

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences. Unless otherwise specified, the setting is for general metric space (X, d).

1. In a general metric space, define

a.) open set:

A set O is open if for each $x_0 \in O$, there is an $r > 0$ so that $N_r(x_0) \subseteq O$.

b.) limit point of a set:

A point p_0 is called a limit point of a set A if each neighborhood of p_0 contains a point from A, different from p_0.

c.) closed set:

A set C is called closed if it contains all its limit points.

2. a.) Using the definitions, prove that a set is closed if and only if its complement is open.

Let F be a set and $E := F^c$. From the notes:

F is closed $\Leftrightarrow$ (F contains all its limit points) $\Leftrightarrow$ (no element of F^c (i.e. of E) is a limit point of F)

$\Leftrightarrow$ (each element of E has a nbhd with no elements from F) $\Leftrightarrow$ (each element of E is an interior pt) $\Leftrightarrow$ E is open

b.) Prove that $[0, 1]$ is a closed subset for the metric space $(\mathbb{R}, |\cdot|)$.

First notice $O_1 := (1, \infty)$ is open (if $x_0 \in O_1$, let $r := x_0 - 1 > 0$) and $O_2 := (-\infty, 0)$ is open (if $x_0 \in O_2$, let $r := -x_0 > 0$), then $[0,1]^c = O_1 \cup O_2$ is open.

3. Prove that the set $N_r(p_0)$ is open.

From the notes: If $p \in N_r(p_0)$, then let $\varepsilon := r - d(p, p_0) > 0$.
If $q \in N_\varepsilon(p)$, then $d(q,p_0) \leq d(q,p) + d(p,p_0) < \varepsilon + d(p,p_0) = r$
and so $N_\varepsilon(p) \subseteq N_r(p_0)$. Hence each point $p \in N_r(p_0)$ is an interior point. ▨

4. Prove the property of products of limits: If in $(\mathbb{R}, |\cdot|)$, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} a_n b_n = ab$

$\{b_n\}$ convergent $\Rightarrow \{b_n\}$ is bnded i.e. $\exists\, M \ni |b_n| \leq M$, all $n \in \mathbb{N}$.
Given $\varepsilon > 0$ let $\varepsilon' := \frac{\varepsilon}{M + |a| + 1}$. $a_n \to a \Rightarrow \exists\, N_1 \ni n \geq N_1$ implies $|a_n - a| < \varepsilon'$. $b_n \to b \Rightarrow \exists\, N_2 \ni n \geq N_2$ implies $|b_n - b| < \varepsilon'$. Set $N := \max(N_1, N_2)$, then $n \geq N$ implies

$$|a_n b_n - ab| \leq |b_n(a_n - a) + a(b_n - b)| \leq |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b|$$
$$\leq M \cdot \varepsilon' + |a| \cdot \varepsilon' = \frac{M + |a|}{M + |a| + 1}\varepsilon < \varepsilon. \quad ▨$$

5. a.) Define **Cauchy** sequence.

$\{a_n\}_{n=1}^{\infty}$ Cauchy means $\forall \varepsilon > 0\ \exists\, N \in \mathbb{N} \ni n, m \geq N$ implies $d(a_n, a_m) < \varepsilon$.

b.) Prove that each convergent sequence is Cauchy.

Suppose $\{a_n\}$ is convergent. Let $\varepsilon > 0$ be arbitary. Apply definition of "convergent" to $\varepsilon' = \varepsilon/2$ to get $N \in \mathbb{N} \ni$
$$d(a_j, a) < \varepsilon', \text{ if } j \geq N.$$
So if $n, m \geq N$, then
$$d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \varepsilon' + \varepsilon' = \varepsilon. \quad ▨$$

6. a) State Bernoulli's inequality.

If $a > 0$, then
$$1 + n \cdot a \leq (1+a)^n, \quad \text{for all } n \in \mathbb{N},$$

b) Apply this to show that the sequence $b_n := \frac{n}{\sqrt{2^n}}$ is bounded.

From part a) $\frac{n \cdot a}{(1+a)^n} \leq 1$ or $\frac{n}{(1+a)^n} \leq \frac{1}{a}$. But $\sqrt{2^n} = (\sqrt{2})^n$ so set $1+a = \sqrt{2}$ or $a := \sqrt{2} - 1 > 0$. This gives
$$0 \leq b_n \leq \frac{1}{\sqrt{2} - 1}$$

c) Using part b), show that $\lim_{n\to\infty} \frac{n}{2^n} = 0$.

$$0 \leq \frac{n}{2^n} = \frac{n}{\sqrt{2^n}} \left(\frac{1}{\sqrt{2}}\right)^n = b_n \cdot r^n \to 0 \quad \text{as } n \to \infty$$
since $r = \frac{1}{\sqrt{2}} < 1$ & $\{b_n\}$ is bounded.

d) Using part c) and other properties of sequences, determine if the following limit exists and if so compute its value:
$$\lim_{n\to\infty} \frac{n^2 + n^3/2^n + 1}{n - 3n^2}.$$

Notice $\frac{n^2 + \frac{n^3}{2^n} + 1}{n - 3n^2} = \frac{\left(1 + \frac{n}{2^n} + \frac{1}{n^2}\right)}{\left(\frac{1}{n} - 3\right)}$. But $\lim_{n\to\infty}\left(1 + \frac{n}{2^n} + \frac{1}{n^2}\right)$ exists by part c) & "limit of sums". This limit is $1 + 0 + 0 = 1$. The denominator has limit $= -3 \neq 0$ so the limit of the quotients exists and equals $\boxed{\frac{1}{-3}}$.

7. Prove that p_0 is a limit point of a set E if and only if there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ from E, with $p_n \neq p_0$ and $\lim_{n\to\infty} p_n = p_0$.

p_0 limit point of E $\Leftrightarrow$ $\forall n \in \mathbb{N}$ $\exists$ $p_n \neq p_0$ so that $p_n \in E$ with $d(p_n, p_0) < \frac{1}{n}$.

& hence $\lim_{n\to\infty} p_n = p_0$, $p_n \neq p_0, \forall n$, $p_n \in E$

The condition $\lim_{n\to\infty} p_n = p_0$ with $p_0 \neq p_n \in E$ means $\forall$ nbhd $N_\varepsilon(p_0)$ $\exists$ an infinite number of elements from $\{p_n\}$, not equal to p_0, which belong to $N_\varepsilon(p_0)$ which says that $p_0 \in E'$.

8. Suppose $\{a_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence which is bounded from below, then prove that it converges and find its limit.

The set $A = \{a_n \mid n \in \mathbb{N}\}$ is bounded from below & is nonempty, so $\alpha := \text{glb}\, A$ exists. By defn of glb for each $\varepsilon > 0$ $\exists\, n_1$ so that

$$\alpha \leq a_{n_1} < \alpha + \varepsilon .$$

Set $N := n_1$ then $n \geq N \Rightarrow \quad \alpha \leq a_n \leq a_N < \alpha + \varepsilon$

so

$$0 \leq a_n - \alpha < \varepsilon ,$$

or

$$|a_n - \alpha| < \varepsilon \quad \text{if } n \geq N. \quad \blacksquare$$

9. (Graduate Student Problem) Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a number $a > 0$. Using properties we have developed for sequences, prove that this implies

$$\lim_{n \to \infty} \sqrt{a_n} = \sqrt{a} \quad .$$

Note: $a_n \to a$ & $a > 0$ implies $\exists\, N_1$ so that $n \geq N_1 \Rightarrow a_n > 0$.
For each $n \geq N_1$

$$(*) \quad \left| \sqrt{a_n} - \sqrt{a} \right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{1}{\sqrt{a}} |a_n - a| .$$

Given $\varepsilon > 0$, set $\varepsilon' := \sqrt{a} \cdot \varepsilon > 0$. Since $a_n \to a$, pick N ($N \geq N_1$) so that $|a_n - a| < \varepsilon'$ if $n \geq N$. Apply this to $(*)$ to get

$$\left| \sqrt{a_n} - \sqrt{a} \right| < \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon'}{\sqrt{a}} = \varepsilon \quad , \text{ if } n \geq N. \quad \blacksquare$$