Connectedness
Handout #6

Defn 1. A *disconnection* of a set A is two nonempty sets A_1, A_2 whose disjoint union is A and each is open relative to A. A set is said to be *connected* if it does not have any disconnections.

Example. The set $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ is disconnected.

Theorem 1. Each interval (open, closed, half-open) I is a connected set.
Proof. Let A_1, A_2 be a disconnection for I. Let $a_j \in A_j, \quad j = 1, 2$. We may assume WLOG that $a_1 < a_2$, otherwise relabel A_1 and A_2. Consider $E_1 := \{x \in A_1 | x \leq a_2\}$, then E_1 is nonempty and bounded from above. Let $a := \sup E_1$. But $a_1 \leq a \leq a_2$ implies $a \in I$ since I is an interval. First note that by the lemma to the least upper bound property either $a \in A_1$ or a is a limit point of A_1. In either case, $a \in A_1$ since A_1 is closed relative to I. Since A_1 is also open relative to the interval I, then there is an $\epsilon > 0$ so that $N_\epsilon(a) \in A_1$. But then $a + \epsilon/2 \in A_1$ and is less than a_2, which contradicts that a is the sup of E_1. $\square$

Theorem 2. If A is a connected set, then A is an interval.
Proof. Otherwise, there would be $a_1 < a < a_2$, with $a_j \in A$ and $a \notin A$. But then $\mathcal{O}_1 := (-\infty, a) \cap A$ and $\mathcal{O}_2 := (a, \infty) \cap A$ form a disconnection of A. $\square$

Theorem 3. The continuous image of a connected set is connected. The continuous image of $[a, b]$ is an interval $[c, d]$ where $c = \min_{a \leq x \leq b} f(x)$ and $d = \max_{a \leq x \leq b} f(x)$.

Proof. Any disconnection of the image $f([a, b])$ can be 'drawn back' to form a disconnection of $[a, b]$: if $\{\mathcal{O}_1, \mathcal{O}_2\}$ forms a disconnection for $f(I)$, then $\left\{f^{-1}(\mathcal{O}_1), f^{-1}(\mathcal{O}_2)\right\}$ forms a disconnection for $I = [a, b]$. $\square$

Corollary 1. (Intermediate Value Theorem) Suppose f is a real-valued function which is continuous on an interval I. If $a_1, a_2 \in I$ and y is a number between $f(a_1)$ and $f(a_2)$, then there exists a between a_1 and a_2 such that $f(a) = y$.
Proof. We may assume WLOG that $I = [a_1, a_2]$. We know that $f(I)$ is a closed interval, say I_1. Any number y between $f(a_1)$ and $f(a_2)$, belongs to I_1 and so there is an $a \in [a_1, a_2]$ such that $f(a) = y$. $\square$

Theorem 4. Suppose that $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point, i.e. there is an $\alpha \in [a, b]$ such that $f(\alpha) = \alpha$.

Proof. Consider the function $g(x) := x - f(x)$, then $g(a) \leq 0 \leq g(b)$. g is continuous on $[a, b]$, so by the Intermediate Value Theorem, there is an $\alpha \in [a, b]$ such that $g(\alpha) = 0$. This implies that $f(\alpha) = \alpha$. $\square$

Note. There are some immediate consequences of these ideas.

- First, we can get a better idea of the structure of general open sets in the real line. Each open subset of $I\!R$ is the countable disjoint union of open intervals. This is seen by looking at open *components* (maximal connected sets) and recalling that each open interval contains a rational. Relatively (with respect to $A \subseteq I\!R$) open sets are just restrictions of these sets.

- Connectedness is the basis of *root finding*: for example with the Bisection method. Consider the example of solving for polynomial roots, or $\sin(x) = x$ in the interval $(0, \infty)$.

- It also permits us to study inverse functions of continuous, strictly monotone functions. We see that the continuous image under a monotone map f of a closed interval $[a, b]$ is a closed interval $[f(a), f(b)]$. That is any continuous strictly monotone increasing function f maps $[a, b]$ one-to-one and onto $[f(a), f(b)]$. (Using compactness in the next notes, we will show that in this settings, inverse functions are also continuous.)