MATH 554/703 I - FALL 08
Lecture Note Set # 3

Introduction to Metric Space Topology

Please read Chapter 2 of Rudin which deals with the concepts of metric space topology (pages 30-36). Some portions are outlined again here, in the order of lecture presentation, for your convenience. An idea of the proof is given in some cases, but the full proofs are given during in class.

Defn. A set X equipped with a non-negative function $d : X \times X \to [0, \infty]$ is called a *metric space* if d satisfies

1. $d(p, q) \geq 0$ for all $p, q \in X$, and $d(p, q) = 0$ if and only if $p = q$.
2. $d(p, q) = d(q, p)$
3. $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Defn. A *neighborhood* of radius r of a point p is defined by

$$N_r(p) := \{q \in X |\ d(p, q) < r\}$$

On the real line $\mathbb{R}$, the neighborhoods of a point a take the form of intervals $(a-r, a+r)$ since the inequality $|a - x| < r$ is equivalent to $-r < x - a < r$, which in turn is equivalent to $a - r < x < a + r$.

Defn. Suppose (X, d) is a metric space.

1. A *limit point* p of a set $E \subseteq X$ is a point for which each of its neighborhoods contains a point from E distinct from p.
2. E' denotes the set of all limit points of a set E and is called its *derived set.*
3. A set E is called *closed* if it contains all its limit points (i.e. $E' \subseteq E$).
4. $\bar{E} := E \cup E'$ denotes the *closure* of a set E.
5. A point p of E is called an *isolated point* if it is not a limit point of E.

Theorem. If p is a limit point of E, then each neighborhood of p contains an infinite number of members of E.
(think of the proof that every interval contains an infinite number of rational numbers, from the result that each interval had at least one)

Corollary. A finite set has no limit points. (i.e. , E finite implies $E' = \emptyset$)

Defn. Suppose (X, d) is a metric space.

1. A point p is called an *interior point* of E if there exists an $r > 0$ so that $N_r(p) \subseteq E$.

2. A set E is called *open* if each point of E is an interior point of E.
3. The set of all interior points is called the *interior of* E and is denoted by E^o. In this case, a set E is open is equivalent to $E \subseteq E^o$

Theorem. In a metric space, each neighborhood is an open set.

Proof. For $q \in B_r(p_0)$, let $\epsilon := r - d(p_0, q)$, then $B_\epsilon(q) \subset B_r(p_0)$.

Corollary. For any set E, the interior of E is an open set.

(*Proof.* If $p \in E^o$, then there is a nhbd $N_r(p) \subseteq E$. From the previous theorem $N_r(p)$ is open and so contains a corresponding nhbd for each of its points. This shows that each point of $N_r(p)$ is an interior point of E.)

Theorem. A set E is open if and only if its complement E^c is closed.

(Essentially a tautology. Using the definitions: each point of E is an interior point $\iff$ each point of E has a nhbd with no elements of E^c $\iff$ no element of E can be a limit point of E^c $\iff$ E^c contains all its limit points.)

Corollary. Let (X, d) be a metric space, then

1. Arbitrary unions of open sets are open.
   (Use an r from any of the open sets of the collection to which the point belongs.)
2. Finite intersections of open sets are open.
   (Use the smallest r for the open sets of the collection to which the point belongs.)
3. Arbitrary intersections of closed sets are closed.
   (Use De Morgan's law and apply the previous Theorem.)
4. Finite unions of closed sets are closed.
   (Use De Morgan's law and apply the previous Theorem.)

Note. Arbitrary intersectons of open sets need not be open.

a.) If $\mathcal{O}_n := (-1/n, 1/n)$, then $\displaystyle\bigcap_{n=1}^{\infty} \mathcal{O}_n = \{0\}$.

b.) If $\mathcal{O}_n := (-1/n, 1 + 1/n)$, then $\displaystyle\bigcap_{n=1}^{\infty} \mathcal{O}_n = [0, 1]$.

c.) If $\mathcal{O}_n := (-1/n, 1)$, then $\displaystyle\bigcap_{n=1}^{\infty} \mathcal{O}_n = [0, 1)$.

Examples, examples, examples....

(For example, $\mathbb{R}, \mathbb{C}, \mathbb{R}^d$, discrete metric, $C[a, b]$ - the space of continuous functions.)