Soln Homework # 5

1. Theorem. If x_0 is a limit point of dom(f) and dom(g), $\lim_{x \to x_0} f(x) = L_1$, $\lim_{x\to x_0} g(x) = L_2$, then $f+g$ has a limit at x_0 and
$$\lim_{x \to x_0} \left(f(x)+g(x)\right) = L_1 + L_2 .$$

proof dom(f+g) = dom(f) $\cap$ dom(g) so $x_0 \in$ dom(f+g).
Given $\varepsilon > 0$, $\exists\ \delta_1 > 0$ so that
$$(1) \qquad |f(x) - L_1| < \varepsilon/2 \quad \text{if } x \neq x_0,\ x \in \text{dom}(f) \text{ and } |x-x_0| < \delta_1$$
For this ε, $\exists\ \delta_2 > 0$ so that
$$(2) \qquad |g(x) - L_2| < \varepsilon/2 \quad \text{if } x \neq x_0,\ x \in \text{dom}(g) \text{ and } |x-x_0| < \delta_2$$
Let $\delta = \min(\delta_1, \delta_2)$, then $\delta > 0$. If $0 < |x-x_0| < \delta$, $x \in$ dom(f+g), then $0 < |x-x_0| < \delta_j$ $(j=1,2)$ and so both (1) and (2) hold. Therefore
$$|(f+g)(x) - (L_1+L_2)| \leq |f(x)-L_1| + |g(x)-L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \blacksquare$$

2. Where is $f(x) = \begin{cases} 3x+2, & x \geq -1 \\ -2x+1, & x < -1 \end{cases}$ continuous?

proof On the open set $(-1, \infty)$, f is a polynomial and so is continuous. Similarly on the interval $(-\infty, -1)$. [polynomials are combinations of sums & products of the identity and constant functions.] To prove f is continuous at $x_0 = -1$
At $x_0 = -1$, $\lim_{x \nearrow x_0} f(x) = \lim_{x \nearrow x_0} (-2x+1) = +3$, but
$$\lim_{x \searrow x_0} f(x) = \lim_{x \searrow x_0} (3x+2) = -1 .$$
so f is not continuous at $x_0 = -1$.

3. This was on Test # 3.