Homework # 4 Solutions
Math 554

#1 Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence. We need to show that it is contained in some nbhd. Apply defn of Cauchy with $\varepsilon := 1 > 0$. $\exists\, N \in \mathbb{N} \ni d(p_n, p_m) < 1$ if $n, m \geq N$. In particular,

$$d(p_N, p_n) < 1 \quad \text{if } n \geq N.$$

Set $d_j := d(p_N, p_j)$ for $j = 1, 2, \dots, N-1$. Then if the nbhd radius R is taken as

$$R := \max\{d_1, d_2, \dots, d_{N-1}\} + 1$$

we will have

$$d(p_n, p_N) \leq R \quad \text{, all } n$$

and so $\{p_n \mid n = 1, 2, \dots\}$ is a bounded set. ∎

#2 From problem # 1, $A := \{a_n \mid n \in \mathbb{N}\}$ is a bounded set so

$$A \subseteq [-M, M] \quad \text{for some } M > 0.$$

Consider $A_N := \{a_j \mid j = N, N+1, \dots\}$, then $A_N \neq \phi$ & bnded from above with

$$\dots \subseteq A_{N+1} \subseteq A_N \subseteq \dots \subseteq A_2 \subseteq A_1 \subseteq A \subseteq [-M, M]$$

From our earlier fact that $(C \subseteq D \subseteq \mathbb{R}) \Rightarrow (\text{lub}\, C \leq \text{lub}\, D)$ [lub D is an upper bound for D & therefore for C] we have that

$$-M \leq \dots \leq \alpha_{N+1} \leq \alpha_N \leq \dots \leq \alpha_2 \leq \alpha_1 \leq +M$$

Therefore if $\alpha := \text{glb}\{\alpha_n \mid n = 1, 2, \dots\}$, then $\lim_{N \to \infty} \alpha_N = \alpha$. ∎

#3 $\{p_n\}_{n=1}^{\infty} \subseteq X$ is Cauchy & p_0 is a limit point of $A := \{p_1, p_2, \dots\}$ as a set. We show $\lim_{n \to \infty} p_n$ exists and equals p_0. Let $\varepsilon > 0$. $\{p_n\}$ Cauchy $\Rightarrow$ $\exists\, N \in \mathbb{N} \ni d(p_n, p_m) < \varepsilon/2$ if $n, m \geq N$. Also $p_0 \in A'$, so $N_{\varepsilon/2}(p_0)$ contains an infinite number of members of A. Since $\{p_1, p_2, \dots, p_{N-1}\}$ is only a finite set, then at least one of this infinite number must come from $p_N, p_{N+1}, p_{N+2}, \dots$. Denote this member p_{n_1}, $n_1 \geq N$. Then if $n \geq N$,

$$d(p_n, p_0) \leq d(p_n, p_{n_1}) + d(p_{n_1}, p_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad ∎$$

HW #4 Solutions (Continued)

#4 (a) Let $b_n := 15$ & $c_n := \frac{1}{n}$. We know $\lim_{n\to\infty} b_n = 15$ & $\lim_{n\to\infty} c_n = 0$, so using the property for limits & products

$$\lim_{n\to\infty} 15/n = (\lim_{n\to\infty} b_n)(\lim_{n\to\infty} c_n) = 15 \cdot 0 = 0.$$

(b) Use problem #5 with $b_n := (-1)^n+1$ & $a_n = \frac{1}{n^2}$. By products of limits ($a_n = \frac{1}{n}\frac{1}{n}$) we know $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Finally $b_n = (-1)^n+1$ satisfies $|b_n| \le 2$. Problem #5 implies $\lim_{n\to\infty} \frac{(-1)^n+1}{n^2}$ exists and equals 0.

(c) Write $a_n = \frac{1}{n}\left(\frac{1-3/n}{5/n^3-3}\right)$. Use $\lim_{n\to\infty}(1-3/n) = 1$ & $\lim_{n\to\infty}(5/n^3-3) = -3 \ne 0$ & apply quotient of limits property to get

$$\lim_{n\to\infty}\left(\frac{1-3/n}{5/n^3-3}\right) = -\frac{1}{3}.$$

Finally, apply product of limits property to complete the problem.

#5 $\{b_n\}$ is bdded so there exist $M>0 \ni |b_n| \le M$, all $n \in \mathbb{N}$. To show $\lim_{n\to\infty} a_n b_n = 0$, let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = 0$, then for $\varepsilon' := \varepsilon/M > 0$ $\exists$ $N \ni n \ge N$ implies

$$|a_n| = |a_n - 0| < \varepsilon' \quad \text{if } n \ge N.$$

But then

$$|a_n b_n - 0| = |a_n b_n| \le |a_n| \cdot M < \varepsilon' \cdot M = \varepsilon$$

if $n \ge N$. done $\blacksquare$

#6 The triangle inequality implies

$$|a| = |(a-b)+(b)| \le |a-b| + |b|.$$

The problem is finished by subtracting $|b|$ from both sides of the inequality.

#7 Done in class, but the basic idea is to use $\varepsilon := |b|/2 > 0$ and reverse Δ-$\le$ to get for $n \ge N$

$$|b| - |b_n| \le |b - b_n| < \frac{|b|}{2}$$ & add $|b_n| - \frac{|b|}{2}$ to both sides $\blacksquare$