Math 554 / 703 I

Graduate Student Problem & Extra Credit

Nov. 1, 2008

__Theorem__ Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence of real numbers, then the sequence converges.

Proof $\{a_n\}_{n=1}^{\infty}$ Cauchy implies it is bounded, so $\exists\, a, b \in \mathbb{R} \ni$

$$a \le a_n \le b \qquad \text{all } n.$$

The set $A_N := \{a_n \mid n \ge N\}$ is non-empty and bnded from above so has a least upper bound (i.e. supremum) which we denote by

$$\alpha_N := \text{lub } A_N.$$

Similarly, define $\beta_N := \text{glb } A_N$. We know $\beta_N \le \alpha_N$ (all N), $\alpha_N \downarrow$, and $\beta_N \uparrow$, so if M, N are arbitrary then

$$a \le \beta_M \le \alpha_N \le b$$

(for example, if $M \ge N$ then $\beta_M \le \alpha_M \le \alpha_N$. Similarly if $M < N$.)

Let $\beta := \text{lub}\{\beta_M \mid M \in \mathbb{N}\}$ & $\alpha := \text{glb}\{\alpha_N \mid N \in \mathbb{N}\}$, then the defn of lub so $\beta \le \alpha_N$, $\forall N$. The defn of glb shows $\beta \le \alpha$. Therefore

(0) $$a \le \beta_M \le \beta \le \alpha \le \alpha_N \le b \qquad \text{all } M \,\&\, N.$$

(Note: α is called $\overline{\lim\limits_{n\to\infty}}\, a_n$ & β is called the $\underline{\lim\limits_{n\to\infty}}\, a_n$.) these exist for any bounded sequence.

For Cauchy sequences we show $\alpha = \beta$ and $\lim\limits_{n\to\infty} a_n = \alpha$:

Let $\varepsilon > 0$ & set $\varepsilon' := \varepsilon/3$. $\{a_n\}$ Cauchy $\Rightarrow \exists\, N \in \mathbb{N} \ni$

(1) $$|a_n - a_m| < \varepsilon' \quad \forall\, n, m \ge N.$$

By defn of α_N & β_N $\exists\, n', m' \ge N$ so that (draw a picture)

(2) $$\alpha_N - \varepsilon' < a_{n'} \le \alpha_N$$

(3) $$\beta_N \le a_{m'} < \beta_N + \varepsilon'$$

Hence

(4) $$0 \le \alpha - \beta \overset{(0)}{\le} \alpha_N - \beta_N \overset{(2)\,\&\,(3)}{<} (a_{n'} + \varepsilon') - a_{m'} + \varepsilon' \le |a_{n'} - a_{m'}| + 2\varepsilon' \overset{(1)}{<} \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so $\alpha = \beta$.
To show $\lim_{n\to\infty} a_n = \alpha$, use the fact that

$$\beta_N \leq \alpha, a_n \leq \alpha_N \quad , \text{if } n \geq N$$

and the 2nd thru 5th inequalities of relation (4) to get

$$|a_n - \alpha| \leq \alpha_N - \beta_N < \varepsilon \qquad \text{if } n \geq N. \blacksquare$$