Math 554/703I - Analysis I
Test 3 – April 14, 2004

Name: ____________________________

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1	(20 *pts*)
2	(10 *pts*)
3	(10 *pts*)
4	(10 *pts*)
5	(15 *pts*)
6	(10 *pts*)
7	(15 *pts*)
8	(10 *pts*)

1. Give an example of each of the following and (very) briefly justify your answer:

   (a) A closed set of real numbers that is not connected.
   One Soln: $\{0, 1\}$ is closed.

   (b) A set of real numbers that is complete but not connected.
   One Soln: $[0, 1] \cup [2, 3]$, a subset of the real numbers is complete iff it is closed.

   (c) A compact set of real numbers that is not connected.
   One Soln: $[0, 1] \cup [2, 3]$ is closed and bounded but is not connected.

   (d) A real-valued continuous function that does not satisfy the Intermediate Value Theorem.
   One Soln: Let $\text{dom}(f) = \{0\} \cup [1, 2]$ and $f(x) = x$ on that domain. The intermediate value of $\frac{1}{2}$ is not attained.

   (e) A real-valued function that is continuous at a point x_0, but is not differentiable.
   One Soln: The function $f(x) = |x|$ is continuous at $x_0 = 0$, but is not differentiable there; observe sequential limits of the difference quotient applied to the sequences $\{-\frac{1}{n}\}$ and $\{\frac{1}{n}\}$.

2. Prove that each compact set is closed.

   Proof: Suppose that x_o be a limit point of the compact set K which does not belong to K. Consider the open sets $\mathcal{O}_n$ which are the complements of the closed balls $C_n = \{x| \, |x - x_0| \leq \frac{1}{n}\}$. Since $x_0 \notin K$, then the collection of $\mathcal{O}_n$'s form an open cover of K and so must have a finite subcover. Since $\mathcal{O}_n \subset \mathcal{O}_{n+1}$, then the largest set in this finite collection will cover K. Call that set $\mathcal{O}_N$. But then $B_{\frac{1}{N}}(x_0) \subset C_N$ is disjoint from K. **Contradiction**, since each $B_\epsilon(x_0)$ must contain an infinite number of members of K.

3. State and sketch a proof of the **Heine-Borel theorem**.
   Soln: See course lecture notes as well as daily lecture notes.

4. a.) Define **connectedness** for a set of real numbers A.
   Soln: See course lecture notes as well as daily lecture notes.
   b.) Prove that if a continuous function f is defined on an interval I, then the range of f is an interval.

   Proof: A subset of real numbers with the standard metric is connected iff it is an interval. Since the continuous image of a connected set is connected, then the continuous image of an interval is an interval.

5. a.) Define **open cover** for a set.
   b.) Define what it means for a set to be **compact**.
   c.) Suppose K is compact and $f : K \to I\!R$ is continuous. Prove that $f[K]$ is compact.

   Soln: See course lecture notes as well as daily lecture notes.

6. State and sketch the proof of the **Extreme Value Theorem**

   Soln: See course lecture notes as well as daily lecture notes.

7. State and derive the Product Rule for differentiation.
   Soln: See course lecture notes as well as daily lecture notes.

8. Using the **definition** of the derivative and properties of limits, prove that when $f(x) := \sqrt{x}$, then

$$f'(x_0) = \frac{1}{2\sqrt{x_0}}$$

Proof: We just need to show the limit

$$\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$$

exists and equals $L = \frac{1}{2\sqrt{x_0}}$. But

$$x - x_0 = (\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})$$

and so

$$\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}}$$

which by the limit theorems for quotients and sums, and the fact that the square root function is continuous at $x_0 > 0$ (and so $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$) implies that

$$\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \frac{1}{2\sqrt{x_0}}.$$