Math 554.01 - Analysis I
Test 2 – March 5, 2004

1	(15 pts)
2	(20 pts)
3	(20 pts)
4	(20 pts)
5	(15 pts)
6	(10 pts)

Name: ______________________ 4 Digit CODE: ______

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1. a.) Define *metric*.

Notes or Text

E is a set & $d: E\times E \to [0,\infty)$ with the properties
(1) $d(p,q)\geq 0$, all $p,q\in E$
$d(p,q)=0 \iff p=q$.
(2) $d(p,q)=d(q,p)$, all $p,q\in E$.
(3) $d(p,q)\leq d(p,r)+d(r,q)$, all $p,q,r\in E$.

b.) Give two examples of metric spaces. (You do not need to verify the properties.)

Notes or text.

Both the set and the specific metric must be provided.

2. Let (E,d) be a metric space.
a.) Define *open set*.

Notes or text

S is open means if $p\in S$ then there exists $\varepsilon>0$ so that $B_\varepsilon(p)\subseteq S$.

b.) Prove that an open ball is an open set.

Suppose $p\in B_r(p_0)$. Let $\varepsilon := r-d(p,p_0)>0$. If $q\in B_\varepsilon(p)$, then $d(p,q)<\varepsilon$ and so
$$d(q,p_0)\leq d(q,p)+d(p,p_0)<\varepsilon+d(p,p_0)=r.$$
therefore $B_\varepsilon(p)\subseteq B_r(p_0)$ & $B_r(p_0)$ is open. ∎

c.) Let d be the discrete metric on a set E. Prove that each subset S of E is a closed set.

It suffices to show each set is open, by considering complements. But each point is open, since $\{p_0\}=B_{1/2}(p_0)$ is open. Arbitrary unions of open sets are open, so every set in (E,d) is open. ∎

3. a.) Give the definition of a closed set.

Notes or text $\quad C$ is closed means the complement of C is open.

b.) Give the definition of a limit point of a set.

Notes or text $\quad p$ is a limit point of S means each open ball of p contains a point of S different from p.

c.) Prove that a set is closed if and only if it contains all its limit points.

Let S' be the set of all limit points of S. Let $O := \complement S$.

($\Rightarrow$) Suppose S is closed & let $p_0 \in S'$. By definition $O = \complement S$ is open. If p_0 is not in S (i.e. $p_0 \in O$), then $\exists\ B_\varepsilon(p_0)$ which misses S. ※ p_0 is a limit point of S. Hence p_0 must belong to S and $S' \subseteq S$.

($\Leftarrow$) It is enough to show O is open. Suppose not, then $\exists\ p_0 \in O$ such that $\forall\ \varepsilon > 0$ $B_\varepsilon(p_0) \not\subseteq O$. That is, $\forall\ \varepsilon > 0$ there is a member of $\complement O = S$ which belongs to $B_\varepsilon(p_0)$. That member cannot be p_0 because $p_0 \in O$. Hence p_0 is a limit point of S. But $S' \subseteq S$, so $p_0 \in S$. ※. Hence O is open & S is closed. ☒

4. Using the definition of "convergence of a sequence," prove that

a.) $\{a_n\}$ converges to a implies that a_n^2 converges to a^2.

(i.e. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that ...)

$\{a_n\}$ convergent implies $\{a_n\}$ is bounded, so $\exists\ M \ni$

$$|a_n| \le M \quad , \quad n = 1, 2, \ldots$$

Let $\varepsilon > 0$, then

$$(*) \quad |a_n^2 - a^2| = |a_n - a||a_n + a| \le |a_n - a|(|a_n| + |a|) \le (M + |a|)|a_n - a|.$$

$a_n \to a$ & $\tilde{\varepsilon} = \varepsilon/(M+|a|+1) > 0$, so $\exists\ N \ni |a_n - a| < \tilde{\varepsilon}$ if $n \ge N$. Hence if $n \ge N$, $(*)$ implies

$$|a_n^2 - a^2| \le (M + |a|) \cdot \tilde{\varepsilon} < \varepsilon.$$

$\therefore \lim_{n\to\infty} a_n^2 = a^2$. ☒

b.) $\{a_n\}$ converges to a implies that $|a_n|$ converges to $|a|$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, then $\exists$ $N \ni n \ge N \Rightarrow$

$$|a_n - a| < \varepsilon.$$

By the reverse triangle inequality

$$\big||a_n| - |a|\big| \le |a_n - a| < \varepsilon$$

if $n \ge N$. Hence $\lim_{n\to\infty} |a_n| = |a|$.

i.e. sums, products, quotients, . . .

5. Using the $\boxed{\text{properties}}$ of limits, determine whether or not the following limit exists. Be sure to state which property you are using as you show your work.

a.) $a_n = 1 - \dfrac{2}{n}$

We know that $a_n = \frac{1}{n} \to 0$ as $n \to \infty$ & $b_n = -2 \to -2$ as $n \to \infty$ so $c_n = -\frac{2}{n} = (-2)(\frac{1}{n}) \to 0$.

$d_n = 1 \to 1$ as $n \to \infty$ so $1 - \frac{2}{n} = d_n + c_n \to 1 - 0 = 1$.

(For this problem an ε-proof is more direct & easier).

b.) $b_n = 2 + \dfrac{3}{n^2}$

Similar to part a) $\left(\frac{1}{n} \to 0\right) \Rightarrow \left(\frac{1}{n^2} = (\frac{1}{n})(\frac{1}{n}) \to 0 \cdot 0\right)$. Therefore by using again products of sequential limits & sums of sequential limits:

$b_n = 2 + \frac{3}{n^2} = 2 + 3(\frac{1}{n})(\frac{1}{n}) \to 2 + 3 \cdot 0 \cdot 0 = 2$

typo corrected during test

c.) Consider the sequence, $c_n = \dfrac{n - 1}{2n^2 + 3}$. Use parts a.) and b.) to determine the convergence of $\{c_n\}$.

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$$C_n = \frac{n-2}{2n^2+3} = \frac{1}{n} \cdot \frac{1 - \frac{2}{n}}{2 + \frac{3}{n^2}} = \frac{1}{n} \frac{a_n}{b_n} \to 0 \cdot \frac{1}{2} = 0, \text{ since } b_n \to 2 \neq 0.$$

using quotients & products of limits.

6. Suppose that E is a metric space and $S \subset E$ is complete. Prove that S is closed.

Suppose S is not closed, then there exists a limit point p of S which is not in S. By our previous work, we know that there exists a sequence $\{p_n\} \subseteq S$ such that $\lim_{n \to \infty} p_n = p$ and $p \in E$ but $p \notin S$. Convergent sequences are Cauchy so $\{p_n\} \subseteq S$ is Cauchy. S is complete so p_n is convergent to a limit in S. Hence $p \in S$. ※ Therefore S must be closed. ∎