MATH 554/703I - ANALYSIS I
TEST 1 – FEBRUARY 6, 2004

1	(15 *pts*)
2	(15 *pts*)
3	(15 *pts*)
4	(15 *pts*)
5	(15 *pts*)
6	(15 *pts*)
7	([illegible] *pts*) 10 pts.
EC	(5 *pts*)

Name: Key

Directions: To receive credit, you must justify your statements, unless the problem is stated otherwise. Answers should be provided in complete sentences. Notation: F denotes a field, '$<$' is an order for an ordered field, $\mathbb{R}$ is the set of real numbers.

1. Suppose F is a field. Prove that $(-1)\cdot a$ is the additive inverse of a

proof We show that

$$(*) \qquad ((-1)\cdot a) + a = 0$$

& use our previous theorem that additive inverses are unique. This gives $(-1)\cdot a = (-a)$. Equation $(*)$ follows from the equations

$$\begin{aligned} ((-1)\cdot a) + a &= ((-1)\cdot a) + (1\cdot a) && (1 \text{ is the multiplicative identity}) \\ &= ((-1)+1)\cdot a && (\text{distributivity property of } F) \\ &= 0\cdot a && ((-1) \text{ is the additive inverse of } 1). \\ &= 0. && (\text{previous theorem stated } `a\cdot 0 = 0'). \end{aligned}$$

2. State and prove the Archimedean principle.

Thm $\forall\ a,b>0\ \ \exists\ n\in\mathbb{N}$ such that $b < n\cdot a$.

proof We use a contrapositive argument: Assume to the contrary that $\forall\, n\in\mathbb{N}\ \ b \geq n\cdot a$. In this case, $b/a \geq n$, $\forall n\in\mathbb{N}$, by multiplying through by $(a^{-1})>0$, and so b/a is an upper bound for $\mathbb{N}$. ※ We have previously shown that $\mathbb{N}$ is not bounded from above (using the Completeness Axiom).

3. a.) For all natural numbers n, use induction to prove that $2^{-n} < \frac{1}{n}$.

p_n: '$2^{-n} < \frac{1}{n}$' is equivalent to the statement '$n < 2^n$' (multiply by $(n \cdot 2^n) > 0$).

Proof is by induction:

<u>n=1</u> p_1: $2^{-1} = 1/2 < 1 = 1/1$ is true. ✓

<u>induction step</u> Assume p_n is true. p_{n+1} is equivalent to the statement $n+1 < 2^{n+1}$.

But $(n+1) < 2^n + 1 < 2^n + n < 2^n + 2^n = 2^n(1+1) = 2^{n+1}$. ∎

b.) Determine the least upper bound and the greatest lower bound of $A := \{2^{-n} \mid n \in N\}$. Justify your answer.

<u>l.u.b.</u> $\ldots < (\frac{1}{2})^{n+1} < (\frac{1}{2})^n < \ldots < (\frac{1}{2})^2 < \frac{1}{2}$ so $\frac{1}{2}$ is an upper bound for A.
If $\gamma < \frac{1}{2}$, then γ is not an upper bound for A since $1/2 \in A$.
There $1/2$ is the least upper bound of A.

<u>g.l.b.</u> $0 < (\frac{1}{2})^n$ for all n so 0 is a lower bound for A. If $\varepsilon > 0$ then $\exists\, n \in \mathbb{N} \ni 0 < \frac{1}{n} < \varepsilon$. By part (a) $0 < 2^{-n} < \frac{1}{n} < \varepsilon$, so $\varepsilon + 0 = \varepsilon$ is not a lower bound for A. Hence 0 is the greatest lower bound of A.

4. Suppose that F is an ordered field:

a.) give the axioms for the positive set P.

(1) if $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$
(2) if $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$
(3) For each $a \in F$, exactly one of the following holds i) $a \in \mathbb{P}$ ii) $a = 0$ iii) $(-a) \in \mathbb{P}$.

b). define what it means for $a < b$ to hold.

$b - a \in \mathbb{P}$

c.) if $a < b$ and $c < 0$, prove that $b \cdot c < a \cdot c$.

$a < b$ is defined as $b - a \in \mathbb{P}$
$c < 0$ - - - - - - $-c \in \mathbb{P}$
} $(-c)(b-a) \in \mathbb{P}$

But $(-c)(b-a) = -bc + ac$. This belonging to $\mathbb{P}$ implies

$bc < ac$. ∎

5. Determine all real numbers x that satisfy

$$x < 3/(x-2)$$

either

Case 1 $(x-2) > 0$

Multiply both sides by the positive number $(x-2)$:

$$(x-2)\cdot x < 3.$$

This is equivalent to $(x^2 - 2x - 3) < 0 \equiv (x-3)\cdot(x+1) < 0$. But this is equivalent to $\left(\text{either } \overset{\text{subcase (a)}}{\begin{pmatrix} x-3<0 \\ \& \; x+1>0 \end{pmatrix}} \text{ or } \overset{\text{subcase (b)}}{\begin{pmatrix} x-3>0 \\ x+1<0 \end{pmatrix}}\right)$, which is equivalent to

$\left(\text{either } (-1 < x < 3) \text{ or } \begin{pmatrix} x>3 \\ \& \; x<-1 \end{pmatrix}\right)$. But in this case 1 x must also be larger than 2, so $\boxed{2 < x < 3}$. $\emptyset$

or

Case 2 $(x-2) < 0$

Multiply again, to get $(x(x-2) > 3) \iff ((x-3)(x+1) > 0) \Rightarrow$ either $\overset{\text{subcase (a)}}{(x>3 \; \& \; x>-1)}$ or $\overset{\text{subcase (b)}}{(x<3 \; \& \; x<-1)}$

$\Rightarrow \boxed{x < -1}$

$\boxed{\text{Soln} = \{x \mid 2 < x < 3 \text{ or } x < -1\}}$

6. Let A be an nonempty subset of $\mathbb{R}$.

a.) Define 'upper bound' for A.

M is an upper bound for A means $\forall a \in A$

$$a \leq M.$$

b.) Define 'least upper bound' for A.

γ is a least upper bound for A means

(1) γ is an upper bound for A

and (2) if M is an upper bound for A, then $\gamma \leq M$.

c.) Prove that least upper bounds are unique.

Let γ_1 & γ_2 be least upper bounds for a set A.

γ_2 is an upper bound & γ_1 is a least upper bound implies

$$\gamma_1 \leq \gamma_2$$

Now reverse the roles of γ_2 & γ_1: γ_1 is an upper bound for A &

γ_2 is a lub for A so

$$\gamma_2 \leq \gamma_1 .$$

Hence $\gamma_1 = \gamma_2$. $\blacksquare$

7. Suppose that $I = (a, b)$ is an open interval and $x_0 \in I$, then prove that there exists an $\epsilon > 0$ so that $(x_0 - \epsilon, x_0 + \epsilon) \subset I$.

proof $a < x_0 < b$. Let $\beta = b - x_0 > 0$ & $\alpha = x_0 - a > 0$. Set ε equal to the smaller of α & β (pick either if they are equal). Then $\varepsilon > 0$ & if $y \in (x_0 - \varepsilon, x_0 + \varepsilon)$, then

$$a = (x_0 - \alpha) \leq (x_0 - \varepsilon) < y < (x_0 + \varepsilon) \leq x_0 + \beta = b.$$

So $\forall\ y \in (x_0 - \varepsilon, x_0 + \varepsilon)$, $y \in (a, b)$. ∎

8. **Extra Credit** Negate the statement:

" for each $\epsilon > 0$ there is a positive number δ such that for every $x \in B_\delta(x_0)$ there holds $|f(x) - f(x_0)| < \epsilon$ "

Statement

$$(\forall \varepsilon > 0)\left(\exists\ \delta > 0 \ni \underbrace{\left((\forall\ x \in B_\delta(x_0)) \Rightarrow (|f(x) - f(x_0)| < \varepsilon)\right)}\right)$$

<u>negative statement</u>

$(\exists \varepsilon_0 > 0)($ is negated $)$

i.e. $(\exists\ \varepsilon_0 > 0)(\forall\ \delta > 0$ (statement is false))

i.e. $(\exists\ \varepsilon_0 > 0)\left(\forall\ \delta > 0\ \left((\exists\ \tilde{x} \in B_\delta(x_0)) \Rightarrow |f(\tilde{x}) - f(x_0)| \geq \varepsilon_0\right)\right)$