MATH 554.01 - ANALYSIS I

TEST 2 – OCTOBER 25, 2001

1	(30 *pts*)
2	(10 *pts*)
3	(10 *pts*)
4	(15 *pts*)
5	(15 *pts*)
6	(15 *pts*)
7	(5 *pts*)

Name: ______________________

Directions: To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1. [**Warmup**] Give an example of each of the following for the metric space of real numbers (you do not need to justify).

   (a) an open set which is not an open interval.

   Examples: $(0,1) \cup (2,3)$; ϕ ; complement of Cantor set

   (b) an infinite closed set which is not a closed interval.

   Examples: $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$; $[0,1] \cup [2,3]$; Cantor set

   (c) a set which is closed, but has no limit points.

   Examples: $\{1\}$; $\{1, \frac{1}{2}\}$, any finite set; ϕ; $\mathbb{N}$

   (d) a set which is open, but has no limit points.

   ϕ

   (e) a sequence which is bounded, but is not convergent.

   Examples: $\{(-1)^n\}_{n=1}^{\infty}$; $\{\sin nx\}_{n=1}^{\infty}$ if $x \neq 0$

   (f) a sequence which is convergent, but is not monotone.

   Example: $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

2. Using the definition of "convergence of a sequence," prove that if $\{b_n\}$ converges to b ($b \neq 0$), then $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.

Suppose $\varepsilon > 0$. Since $b_n \to b$ as $n \to \infty$, then $\exists\, N_1 \in \mathbb{N}$ so that

(1) $|b_n - b| < \frac{|b|^2}{2}\varepsilon$, if $n \geq N_1$.

This is possible since $|b| > 0$. Also since $b_n \to b$ & $b \neq 0$, we have proved that $\exists\, N_2$ so that

(2) $\frac{|b|}{2} < |b_n|$ if $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n b|} \underset{\left[\substack{n \geq N \Rightarrow \\ n \geq N_2 \text{ \& } \\ \text{② holds}}\right]}{<} \frac{|b_n - b|}{\frac{|b|}{2}|b|} = \frac{2}{|b|^2}|b_n - b| \underset{\left[\substack{n \geq N \Rightarrow \\ n \geq N_1 \text{ and} \\ \text{① holds}}\right]}{<} \frac{2}{|b|^2}\left(\frac{|b|^2}{2}\varepsilon\right) = \varepsilon. \quad \blacksquare$$

3. Using the **properties** of limits, determine whether or not the following limit exists. Be sure to state which property you are using as you show your work.

$$\lim_{n \to \infty} \frac{1 + \sqrt{n}}{3 - n}.$$

First rewrite $\frac{1+\sqrt{n}}{3-n} = \frac{\frac{1}{\sqrt{n}} + \frac{1}{n}}{\frac{3}{n} - 1}$. If a_n is the numerator & b_n is the denominator, then $\frac{1}{n} \to 0$ as $n \to \infty$ so by the sum and product rules,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3}{n} - 1 = 3 \cdot \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 1 = 3 \cdot 0 - 1$$

$$= -1 \neq 0.$$

We also know that $\lim_{n\to\infty} \sqrt{\frac{1}{n}} = 0$, so again by the sum rule

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} + \lim_{n\to\infty} \frac{1}{n} = 0 + 0 = 0.$$

By the quotient rule

$$\lim_{n\to\infty} \frac{1+\sqrt{n}}{3-n} = \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{0}{-1} = 0.$$

4. a.) Give the definition of an **open** ϵ**-neighborhood** of a real number x_0.

$$\{x\in X \mid |x-x_0|<\varepsilon\} =: N_\varepsilon(x_0) \quad \text{or} \quad (x_0-\varepsilon, x_0+\varepsilon)\cap X$$

$X =$ space under consideration

b.) Give the definition of an **open** set of real numbers.

A set O is <u>open</u> means $\forall\, x_0\in O\ \exists\ \varepsilon>0 \ni N_\varepsilon(x_0)\subseteq O$.

c.) Prove that intersection of a finite number of open sets is open.

Let $O_1, O_2, \ldots, O_n$ be open sets & let $x_0\in O := \bigcap_{j=1}^{n} O_j$.

$\forall j \quad x_0\in O_j$ & O_j open $\Rightarrow \exists\ \varepsilon_j>0 \ni N_{\varepsilon_j}(x_0)\subseteq O_j$.

Set $\varepsilon = \min_{1\le j\le n} \varepsilon_j$, then $\varepsilon>0$ and for each $1\le j\le n$

$$N_\varepsilon(x_0)\subseteq N_{\varepsilon_j}(x_0)\subseteq O_j .$$

$$\therefore \quad N_\varepsilon(x_0)\subseteq \bigcap_{j=1}^{n} O_j = O. \quad \blacksquare$$

5. a.) Define **limit point** for a set C of real numbers.

x_0 is a <u>limit point</u> for C means $\forall \varepsilon>0\ \exists\ x\in N_\varepsilon(x_0)\cap C$ and $x\neq x_0$.

b.) Define "limit of a function at a point x_0."

$\lim_{x\to x_0} f(x) = L$ means x_0 is a limit point of the domain of f and $\forall\ \varepsilon>0\ \exists\ \delta>0 \ni$ if $|x-x_0|<\delta$ & $x\neq x_0$ & $x\in \mathrm{dom} f$ then $|f(x)-L|<\varepsilon$.

c.) Using the definition, prove that $\lim_{x\to\frac{1}{4}} \sqrt{x} = \frac{1}{2}$.

By algebra note that

$$(*)\quad |f(x)-L| = \left|\sqrt{x}-\tfrac{1}{2}\right| = \left|\frac{x-\frac{1}{4}}{\sqrt{x}+\frac{1}{2}}\right| < \frac{|x-\frac{1}{4}|}{\frac{1}{2}} = 2\left|x-\tfrac{1}{4}\right| .$$

So given $\varepsilon>0$, we set $\delta = \min(\frac{\varepsilon}{2}, \frac{1}{4})$, then $\delta>0$ & $0<|x-\frac{1}{4}|<\delta \Rightarrow \begin{cases} x>0 \text{ \& so } x\in \mathrm{dom}(f) \\ 0<|x-\frac{1}{4}|<\frac{\varepsilon}{2} \Rightarrow (*) \text{ is true.} \end{cases}$ $\blacksquare$

6. a.) Give the definition for a function f to be continuous at a point x_0.

f is continuous at x_0 means

either ① x_0 is an isolated point of dom f

or ② x_0 is a limit point of dom f and

$$\lim_{x \to x_0} f(x) = f(x_0).$$

b.) State an equivalent condition (involving sequences) in order to verify that a function is continuous at x_0.

For each sequence $\{x_n\} \subseteq$ dom f, if $x_n \to x_0$ as $n \to \infty$

then $f(x_n) \to f(x_0)$ as $n \to \infty$.

c). Using properties of limits and part b), show that $f(x) = \dfrac{x^2+1}{\sqrt{x}+2}$ is continuous at $x_0 = 2$.

To show continuity at $x_0 = 2$, we let $x_n \to 2$ as $n \to \infty$. By products & sums of limits we know

$$\lim_{n\to\infty} (x_n^2 + 1) = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} x_n) + 1 = x_0^2 + 1.$$

By the property that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x_0}$ and 'sum of limits' property we also know

$$\lim_{n\to\infty} (\sqrt{x_n} + 2) = \sqrt{x_0} + 2.$$

Apply the quotient property

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n^2+1}{\sqrt{x_n}+2} = \frac{\lim_{n\to\infty}(x_n^2+1)}{\lim_{n\to\infty} \sqrt{x_n}+2} = \frac{x_0^2+1}{\sqrt{x_0}+2} = f(x_0).$$

7. Negate the statement that a function is continuous at a point.

Negating 6 a) give the statement

[x_0 is not an isolated pt of dom(f)] and ["x_0 is a limit pt of dom f & $\lim_{x\to x_0} f(x) = L$" is false], so

x_0 is a limit point of dom f but "$\lim_{x\to x_0} f(x) = L$" is false.

The negation of 5(b) says $\exists\, \varepsilon_0 > 0 \ni \forall\, \delta > 0\ \exists\, x$ so that $x \in$ dom(f), $0 < |x - x_0| < \delta$, but $|f(x) - L| \geq \varepsilon_0$.