Math 554 – Riemann Integration

Handout #9a (Nov. 29)

Defn. A collection of $n+1$ distinct points of the interval $[a,b]$

$$P := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}$$

is called a *partition* of the interval. In this case, we define the *norm* of the partition by

$$\|P\| := \max_{1\le i\le n} \Delta x_i.$$

where $\Delta x_i := x_i - x_{i-1}$ is the *length* of the i-th subinterval $[x_{i-1}, x_i]$.

Defn. For a given partition P, we define the *Riemann upper sum* of a function f by

$$U(f,P) := \sum_{i=1}^{n} M_i \, \Delta x_i$$

where M_i denotes the supremum of f over each of the subintervals $[x_{i-1}, x_i]$. Similarly, we define the *Riemann lower sum* of a function f by

$$L(f,P) := \sum_{i=1}^{n} m_i \, \Delta x_i$$

where m_i denotes the infimum of f over each of the subintervals $[x_{i-1}, x_i]$. Since $m_i \le M_i$, we note that

$$L(f,P) \le U(f,P).$$

for any partition P.

Defn. Suppose P_1, P_2 are both partitions of $[a,b]$, then P_2 is called a *refinement of* P_1, denoted by

$$P_1 \prec P_2,$$

if as sets $P_1 \subseteq P_2$.

Note. If $P_1 \prec P_2$, it follows that $\|P_2\| \le \|P_1\|$ since each of the subintervals formed by P_2 is contained in a subinterval arising from P_1.

Lemma. If $P_1 \prec P_2$, then

$$L(f,P_1) \le L(f,P_2).$$

and

$$U(f,P_2) \le U(f,P_1).$$

Proof. Suppose first that P_1 is a partition of $[a,b]$ and that P_2 is the partition obtained from P_1 by adding an additional point z. The general case follows by induction, adding one point at at time. In particular, we let

$$P_1 := \{x_0 = a < x_1 < \cdots < x_{i-1} < x_i < \cdots < b =: x_n\}$$

and

$$P_2 := \{x_0 = a < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < b =: x_n\}$$

for some fixed i. We focus on the upper Riemann sum for these two partitions, noting that the inequality for the lower sums follows similarly. Observe that

$$U(f, P_1) := \sum_{j=1}^{n} M_j \, \Delta x_j$$

and

$$U(f, P_2) := \sum_{j=1}^{i-1} M_j \, \Delta x_j + M(z - x_{i-1}) + \tilde{M}(x_i - z) + \sum_{j=i+1}^{n} M_j \, \Delta x_j$$

where $M := \sup_{[x_{i-1},z]} f(x)$ and $\tilde{M} := \sup_{[z,x_i]} f(x)$. It then follows that $U(f, P_2) \le U(f, P_1)$ since

$$M, \tilde{M} \le M_i. \quad \square$$

Defn. If P_1 and P_2 are arbitrary partitions of $[a, b]$, then the *common refinement* of P_1 and P_2 is defined as the formal union of the two.

Corollary. Suppose P_1 and P_2 are arbitrary partitions of $[a, b]$, then

$$L(f, P_1) \le U(f, P_2).$$

Proof. Let P be the common refinement of P_1 and P_2, then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2). \quad \square$$

Defn. The *lower Riemann integral* of f over $[a, b]$ is defined to be

$$\underline{\int_a^b} f(x)dx := \sup_{\substack{\text{all partitions} \\ \text{P of [a,b]}}} L(f, P).$$

Similarly, the *upper Riemann integral* of f over $[a, b]$ is defined to be

$$\overline{\int_a^b} f(x)dx := \inf_{\substack{\text{all partitions} \\ \text{P of [a,b]}}} U(f, P).$$

By the definitions of least upper bound and greatest lower bound, it is evident that for any function f there holds

$$\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx.$$

Defn. A function f is *Riemann integrable over* $[a, b]$ if the upper and lower Riemann integrals coincide. We denote this common value by $\int_a^b f(x)\, dx$.

Theorem. A necessary and sufficient condition for f to be Riemann integrable is given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$(*) \qquad U(f, P) - L(f, P) < \epsilon.$$

Note that in this case, the unique number between these two values is $\int_a^b f(x)\,dx$.

Proof. First we show that (*) is a sufficient condition. This follows immediately, since for each $\epsilon > 0$ that there is a partition P such that (*) holds,

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(f,P) - L(f,P) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, then the upper and lower Riemann integrals of f must coincide.

To prove that (*) is a necessary condition for f to be Riemann integrable, we let $\epsilon > 0$. By the definition of the upper Riemann integral as a infimum of upper sums, we can find a partition P_1 of $[a,b]$ such that

$$\int_a^b f(x)dx \leq U(f,P_1) < \int_a^b f(x)dx + \epsilon/2$$

Similarly, we have

$$\int_a^b f(x)dx - \epsilon/2 < L(f,P_2) \leq \int_a^b f(x)dx.$$

Let P be a common refinement of P_1 and P_2, then subtracting the two previous inequalities implies,

$$U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_2) < \epsilon. \qquad \square$$