Compactness
Handout #6

Defn. Suppose that $K \subseteq I\!R$. A collection $\mathcal{G}$ of open subsets such that

$$K \subseteq \bigcup_{\mathcal{O} \in \mathcal{G}} \mathcal{O}.$$

is called an *open cover* of K. K has a *finite subcover* from $\mathcal{G}$ if there exist $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ in $\mathcal{G}$ for which

$$K \subseteq \bigcup_{j=1}^{n} \mathcal{O}_j.$$

Defn. K is called *compact*, if each open cover $\mathcal{G}$ of K has a finite subcover.

Theorem. The continuous image of a compact set is compact.
Proof. Suppose $f : K \to I\!R$ is continuous and K is compact. Each open cover $\mathcal{C}$ of $f[K]$ can be drawn back to an open cover $\tilde{\mathcal{C}}$ of K, by considering the sets

$$\tilde{\mathcal{O}} := f^{-1}(\mathcal{O}), \ \mathcal{O} \in \mathcal{C}.$$

K compact implies that we may draw a finite subcover from $\tilde{\mathcal{C}}$. Each of these members is the inverse image (under f) from a member of $\mathcal{C}$. These form the desired subcover of $f[K]$. □

Theorem. (Heine-Borel) Suppose that $a \leq b$, then the interval $[a, b]$ is compact.
Proof. Let $\mathcal{C}$ be an open cover for $[a, b]$ and consider the set

$$A := \{a \leq x \leq b + 1 \mid [a, x] \text{ has an open cover from } \mathcal{C}\}.$$

Note that A is bounded and nonemtpy (since $a \in A$). Let $\gamma := \text{lub}(A)$. It is enough to show that $\gamma > b$, since if $x_1 \in A$ and $a \leq x \leq x_1$, then $x \in A$. Suppose instead that $\gamma \leq b$, then there must be some $\mathcal{O}_0 \in \mathcal{C}$ such that $\gamma \in \mathcal{O}_0$. But $\mathcal{O}_0$ is open, so there exists $\delta > 0$ so that $B_\delta(\gamma) \subseteq \mathcal{O}_0$. Since γ is the least upper bound for A, then there is an $x \in A$ such that $\gamma - \delta < x \leq \gamma$. But $x \in A$ implies there are members $\mathcal{O}_1, \ldots, \mathcal{O}_n$ of $\mathcal{C}$ whose union covers $[a, x]$. The collection $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n$ covers $[a, \gamma + \delta/2]$. Contradiction, since γ is the least upper bound for the set A. □

Theorem. Each closed subset C of a compact set K is compact.
Proof. Let $\tilde{\mathcal{G}}$ be an open cover for C. Let $\mathcal{O}_0$ be the complement of C, then $\mathcal{O}_0$ is open and $\mathcal{G} := \tilde{\mathcal{G}} \cup \{\mathcal{O}_0\}$ is an open cover for K. There is a finite subcover of

$\mathcal{G}$ which covers K and hence C. This subcover (dropping $\mathcal{O}_0$ if it appears) is the desired finite subcover for C. $\square$

Defn. Suppose $\{a_n\}$ is a sequence. A sequence $\{b_k\}$ is called a *subsequence* of $\{a_n\}$ if there exists a strictly increasing sequence of natural numbers

$$n_1 < n_2 < \ldots < n_k < \ldots$$

such that $b_k = a_{n_k},\ k = 1, 2, \ldots$

Theorem. Suppose that $K \subseteq I\!R$, then TFAE:

a.) K is compact,

b.) K is closed and bounded,

c.) each sequence in K has a subsequence which converges to a member of K,

d.) (Bolzanno-Weierstrass) each infinite subset of K has a limit point in K.

Proof. <u>$(a) \Rightarrow (b)$:</u> To show that K is bounded, consider the open cover of K consisting of the collection of nested open intervals $\mathcal{O}_n := (-n, n),\ n \in I\!N$. To show that K is closed, let x_0 be a limit point of K. Assume to the contrary that $x_0 \notin K$. Consider the open cover of K consisting of the collection of nested open sets $\mathcal{O}_n := \{x \in I\!R \| x - x_0| > 1/n\},\ n \in I\!N$. Any finite subcollection which would cover K would have union whose complement would be a neighborhood of x_0 not intersecting K. This shows that x_0 could not be a limit point of K.
<u>$(b) \Rightarrow (d)$:</u> We use the 'divide and conquer' method, better known as the 'bisection' method. Let A be an infinite subset of K. Since K is bounded, there is an interval $[a, b]$ such that $K \subseteq [a, b]$. Inductively define the closed subintervals as follows. Let $[a_0, b_0] := [a, b]$. Either the left or right half of $[a_0, b_0]$ contains an infinite number of members of K. In the case that it is the right half, set $[a_1, b_1] := [(b_0 + a_0)/2, b_0]$. Set $[a_1, b_1]$ equal to the left half of $[a_0, b_0]$ otherwise. Inductively, let $[a_{n+1}, b_{n+1}]$ be the half of $[a_n, b_n]$ which contains an infinite number of members of A. Notice that the length of this interval is $(b - a)/2^{n+1}$, that the a_n's satisfy $a_n \leq a_{n+1} \leq \ldots < b$ and so must converge to some real number $a \leq x_0 \leq b$. Each neigborhood of x_0 will contain one of the intervals $[a_n, b_n]$ and hence will contain an infinite number of members of A, i.e. x_0 is a limit point of A. This also shows that x_0 is a limit point of the closed set K and must therefore belong to K.
<u>$(d) \Rightarrow (c)$:</u> Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. If the sequence's image is finite, then we may construct a constant subsequence which has the value which we may choose as any of the values of $\{x_n\}_{n=1}^{\infty}$ which is repeated infinitely often. Otherwise, let A

be the range of the sequence. Then A is an infinite subset of K. By the Bolzanno-Weierstrass property, A must have a limit point (x_0 say) which belongs to K. For each $k \in I\!N$, we may find an integer n_k larger than those previously picked (i.e., $n_1, \ldots, n_{k-1}$), so that $|x_{n_k} - x_0| < 1/k$. This is the desired subsequence.
$\underline{(c) \Rightarrow (b) :}$ If K were not bounded, then there would exist a sequence $x_n \in K$ such that $|x_n| > n$. If this sequence had a subsequence which converged, then it would have to be bounded. But each subsequence of $\{x_n\}$ is clearly unbounded. To show that K is closed, we let x_0 be a limit point of K which is not in K. We can then find a sequence $\{x_n\}$ from K which converges to x_0. By condition (c), this has to have a subsequence which converges to a member of K. Contradiction. Each subsequence of a convergent sequence converges to the same limit, in this case x_0, which does not belong to K. □

Corollary. Each continuous function f on a compact set K is bounded.
Proof. The set $f(K)$ is compact and is therefore bounded. □

Corollary. (Extreme Value Theorem) Each continuous function on a compact set attains its maximum (resp. minimum).
Proof. The set $f(K)$ is compact and is therefore bounded and closed. Hence the least upper bound γ for $f(K)$ must belong to $f(K)$. Therefore, there is an $x_0 \in K$ such that $\gamma = f(x_0)$ and so

$$f(x) \leq f(x_0), \text{ for all } x \in K.$$

Similarly, the greatest lower bound of $f(K)$ is attained by some member of K. □

Defn. A function f is called *uniformly continuous* if for each $\epsilon > 0$, $\exists\, \delta > 0$ such that whenever $x_1, x_2 \in \text{dom}(f)$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.

Corollary. Each continuous function on $[a, b]$ is uniformly continuous.
Proof. Suppose not, then negating the definition implies that there exist an $\epsilon_0 > 0$ such that for each $n \in I\!N$ we can find $x_n, y_n \in K$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. K is compact so we can find a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges to some x_0 belonging to K. Notice that $\{y_{n_k}\}_{k=1}^{\infty}$ also converges to x_0 (use an $\epsilon/2$ proof). But f is continuous at x_0, so

$$\epsilon_0 \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| \to 0 \text{ as } \; k \to \infty$$

which is a contradiction. □