Math 554

Handout #5 – Updated for Spring 2004

Defn. A point x_0 is called a *limit point* of a set A if each nbhd of x_0 contains a member of A different from x_0, i.e. for each $\epsilon > 0$, $\quad (N_\epsilon(x_0)\backslash\{x_0\}) \cap A \neq \emptyset$.

Defn. A point $x_0 \in A$ is called an *isolated point of* A if x_0 belongs to A but is not a limit point.

Theorem. A set F is closed if and only if it contains all its limit points.

Theorem. x_0 is a limit point of a set A if and only if there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x_0$, but $x_n \neq x_0, (\forall n \in I\!N)$.

Defn. Suppose that x_0 is a **limit point** of the domain of a function f, then f is said to have *a limit* L *as* $x \to x_0$ if,

$$\forall \epsilon > 0, \exists \delta > 0 \;\ni\; (x \in dom(f) \;\&\; 0 < |x - x_0| < \delta) \Longrightarrow |f(x) - L| < \epsilon.$$

In this case, we use the notation,

$$\lim_{x \to x_0} f(x) = L.$$

Defn. Suppose $A, B \subseteq I\!R$ and $f\colon A \to B$. If $x_0 \in A$, then f is said to be *continuous at* x_0 if for each $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in A$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Note. Remember that a point $x_0 \in A$ is either an isolated point of A or it is a limit point of A. Considering these two cases separately, the definition for continuity of f at a point can be seen to be equivalent to the following in each of the respective situations:

1. if x_0 is an isolated point of A, then f is automatically continuous.

or

2. if x_0 is a limit point of the domain A, then the condition $\lim_{x \to x_0} f(x) = f(x_0)$ must hold.

Defn. Consider a set $B \subseteq I\!R$. A set $\tilde{O} \subseteq B$ is called *open relative to* B (or briefly, *relatively open*) if $\tilde{O} = \mathcal{O} \cap B$ for some open set $\mathcal{O} \subseteq I\!R$. That is to say, $\tilde{O}$ is just the restrict to B of an open set in the real numbers.

Theorem. Suppose that $f\colon A \to B$, where $A, B \subseteq I\!R$, then TFAE (The Following Are Equivalent):

a.) f is continuous at each point of its domain,

b.) for each sequence $x_n \to x_0$, then $f(x_n) \to f(x_0)$ must hold,

c.) For each limit point x_0 of the domain A, $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$.

d.) if $f^{-1}[\mathcal{O}]$ is open for each open subset $\mathcal{O}$ of B.

Note. The same proof above for continuity at a point x_0 can be used to show the corresponding result for limits holds. The only difference is that f is not required to be defined at x_0. The statement reads as:

Suppose that $f\colon A \to B$ is a real-valued function of a real variable, i.e. $A, B \subseteq \mathbb{R}$. If x_0 is a limit point of the domain of f, then TFAE :

a.) $\lim_{x \to x_0} f(x) = L$,

b.) For every sequence $\{x_n\}$ in the domain of f, if $x_n \to x_0$, then $f(x_n) \to L$.

Corollary. The finite sum, product, or the quotient of continuous functions is each continuous on their respective domains.

Corollary. All polynomials are continuous. Rational functions are continuous on their domains.

Theorem. The composition of continuous functions is continuous.

Examples. Each of the following are examples of continuous functions:

1. $f(x) := |x|$
2. $g(x) := \sqrt{x}$
3. $F(x) := \sqrt{\dfrac{x^2 - 2x + 5}{x^3 - 1}}$