Math 554- 703 I - Analysis I
Soln of HW#5 Extra Credit

Theorem. Suppose that S is a set and S' is the collection of all limit points of S. The set $C := S \cup S'$ is called the closure of the set S. Prove that

1. C is closed.
2. If D is a closed set containing S, then $C \subset D$.
3. The closure of S is the intersection of all closed sets which contain S.

Proof. Let $\mathcal{O}$ be the complement of C. To show that C is closed it is same to prove that $\mathcal{O}$ is open. Suppose not. Then there is an element $p_0 \in \mathcal{O}$ such that each ball about p_0 contains members from C. We show that $p_0 \in S' \subset C$ which is a contradiction, since $p_0 \in \mathcal{O}$. Let $\epsilon > 0$. $B := B_\epsilon(p_0)$ contains a member of C, say q. But B is an open set, so there is a ball $\tilde{B} := B_\eta(q) \subset B$. Now $q \in C$, so if $q \in S$, we will be done. If q instead belongs to S', then there is a member of S different than q which belongs to $\tilde{B}$ and hence to B. We have shown in every case that each such ball B of p_0 contains a member of S different than p_0. Hence p_0 is a limit point of S, i.e. a member of S' which is contained in C. This is the contradiction we are looking for.

To prove part (2), if D is closed and contains S, then any limit point of S is a limit point of D. D contains all its limit points, so $S' \subset D$.

To prove part (3), we first use part (2). Each closed set D containing S contains C, so C is contained in the intersection of all such closed sets. C itself is a closed set containing S, so it must contain the intersection of all such closed sets which contain S. $\diamond$