MATH 554/703I - HW #4

Solutions.

No.

Date

1. **Proof**: ① By the definition of $d(p,q) := \begin{cases} 0, & \text{if } p=q \\ 1, & \text{otherwise} \end{cases}$

$d(p,q)$ is always ≥ 0 and

$d(p,q)=0 \iff p=q$

② For $p=q$, $d(p,q)=0=d(q,p)$

$p\neq q$, $d(p,q)=1=d(q,p)$

$\therefore d(p,q)=d(q,p)$

③ Suppose $\exists\ p,q,r \in E$ s.t.

$d(p,q) > d(p,r)+d(r,q)$

then $d(p,q)=1 > 0=0+0=d(p,r)+d(r,q)$

$\therefore d(p,r)=d(r,q)=0$

$\therefore p=q=r \quad \therefore d(p,q)=0$

Contradiction with $d(p,q)=1$.

$\therefore \forall\ p,q,r\in E$

$d(p,q)\leq d(p,r)+d(r,q)$

From ①②③, we know (E,d) is a metric space.

2. Proof: ① $d(p,q)=\phi(|p-q|)$

$= \dfrac{|p-q|}{1+|p-q|} \geq 0$ since $|p-q|\geq 0$

and $d(p,q)=\dfrac{|p-q|}{1+|p-q|}=0 \iff |p-q|=0 \iff p=q$

② $d(p,q)=\dfrac{|p-q|}{1+|p-q|}=\dfrac{|q-p|}{1+|q-p|}=d(q,p)$ since $|q-p|=|p-q|=\pm(p-q)$

③ $\forall\ p,q,r\in E \quad d(p,q)=\dfrac{|p-q|}{1+|p-q|} \quad d(p,r)=\dfrac{|p-r|}{1+|p-r|}$

Directly $d(r,q)=\dfrac{|r-q|}{1+|r-q|}$ Let $a=|p-q|$ $b=|p-r|$, $c=|r-q|$

$\therefore a\leq b+c$ by triangle "$\leq$"

$\therefore a \leq b+c+2bc+bca$

$\therefore a+bca \leq b+2bc+2bca+c$

$\therefore a(1+b+c+bc)\leq b+ba+2bc+2bca+c+ca$

i.e $a(1+c)(1+b)\leq [b+2bc+c](1+a)$

i.e $\dfrac{a}{1+a}\leq \dfrac{b+2bc+c}{1+b+c+bc} \Rightarrow \dfrac{a}{1+a}\leq \dfrac{b}{1+b}+\dfrac{c}{1+c}$

No.

Date

$\therefore \quad d(p,q) \le d(p,r) + d(r,q)$

$\therefore$ By ① ② ③, $d(p,q)$ is a metric.

~~or~~ using that $0 \le \phi(x) \uparrow$ & $\phi(a+b) \le \phi(a) + \phi(b)$ on $[0,\infty]$

where $\phi(x) = \frac{x}{1+x}$ $\Rightarrow d(p,q) = \phi(|p-q|) \le \phi(|p-r|+|r-q|) \le \phi(|p-r|) + \phi(|r-q|)$
$= d(p,r) + d(r,q)$

3. Proof ① $\because |x_1 - y_1| \ge 0$ and $|x_2 - y_2| \ge 0$

$\therefore |x_1 - y_1| + |x_2 - y_2| \ge 0$

$\vec{x} = \vec{y} \Leftrightarrow x_1 = y_1,\ x_2 = y_2 \Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

② $d(\vec{x}, \vec{y}) = \sum_{j=1}^{2} |x_j - y_j| = \sum_{j=1}^{2} |y_j - x_j| = d(\vec{y}, \vec{x})$

③ Let $\vec{z} = (z_1, z_2)$

$d(\vec{x}, \vec{y}) = |x_1 - y_1| + |x_2 - y_2|$

$\le |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$

$= d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y})$

For $\mathbb{R}^d$, $d(\vec{x}, \vec{y}) = \sum_{j=1}^{d} |x_j - y_j|$

4 Proof: $d(f,g) = \text{lub}\{|f(x) - g(x)| : a \le x \le b\}$

① $|f(x) - g(x)| \ge 0 \Rightarrow \text{lub}\,|f(x) - g(x)| \ge 0$

And $\text{lub}\,|f(x) - g(x)| = 0 \Leftrightarrow |f(x) - g(x)| = 0 \Leftrightarrow f(x) = g(x)$

for $a \le x \le b$. $\therefore d(f,g) = 0 \Leftrightarrow f = g$

② $\because |f(x) - g(x)| = |g(x) - f(x)|$

$\therefore \text{lub}\,|f(x) - g(x)| = \text{lub}\,|g(x) - f(x)|$

$\therefore d(f,g) = d(g,f)$

③ $\because |f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)|$ for $\forall f, g, h$. $\forall x \in [a,b]$

$\therefore |f(x) - g(x)| \le \text{lub}\,(|f(x) - h(x)| + |h(x) - g(x)|)$

$\le \text{lub}\,|f(x) - h(x)| + \text{lub}\,|h(x) - g(x)|$

$\therefore |f(x) - g(x)|$ has $\text{lub}\,|f(x) - h(x)| + \text{lub}\,|h(x) - g(x)|$ as upper bound

$\therefore \text{lub}\,|f(x) - g(x)| \le \text{lub}\,|f(x) - h(x)| + \text{lub}\,|h(x) - g(x)|$

$\therefore d(f,g) \le d(f,h) + d(h,g)$