Solutions - HW-3. Math 554

No.

Date

Solutions - HW-3 Math 554

*1). Prove the equivalence statement for "greatest lower bounds": If l is the greatest lower bound for a set A. then

1). l is a lower bound for A

2). If $\varepsilon > 0$, then there exist a in A such that $a < l + \varepsilon$

Proof:

① From the definition of g.l.b. l is clearly a lower bound for A.

② If (2)) not. Then $\exists\ \varepsilon_0 > 0$ for $\forall\ a \in A$ s.t $a \geq l + \varepsilon_0$. So $l + \varepsilon_0$ is a lower bound for A and $l + \varepsilon_0 > l$. This contradicts with the definition of "greatest lower bound". So 2) is right.

*2). Prove that if L_1 and L_2 are both least upper ~~bounds for~~ a set A, then $L_1 = L_2$.

Proof: Since L_1 is the least upper bounds for A. then it is certainly the upper bound for A, and L_2 is the least upper bound for A, therefore $L_2 \leq L_1$.

Similarly, we can prove $L_2 \geq L_1$

So. $L_1 = L_2$

*3). Using the property that the real numbers satisfy the completeness axiom. prove that $\mathbb{R}$ satisfies the property that each nonempty set B which is bounded from below must have a greatest lower bound.

Suppose $A \subseteq \mathbb{R}$ is nonempty and that M is a lower bound for A. then A has a greatest lower bound λ.

No.

Date . .

Proof : Let $B = \{ b \mid b = -a$, for some $a \in A \}$. Let $L = -m$, then L is an upper bound for B. If $b \in B$, then $\exists\, a \in A$ s.t. $b = -a$. But $m \leq a$, so $-a \leq -m = L$. $B \neq \phi$ since $A \neq \phi$. Therefore B has a least upper bound, call it γ. Let $\lambda = -\gamma$. then λ is a lower bound for A since γ is an upper bound for B ($b \leq \gamma,\ \forall b \in B$) $\Leftrightarrow$ ($\lambda \leq a$, $\forall a \in A$). If m is any lower bound for A, then $-m$ is an upper bound for B. γ is the least upper bound $\Rightarrow \gamma \leq -m$, But, $m \leq -\gamma = \lambda$. Therefore, λ is the largest lower bound.

*4). Using induction, prove that $1 + r + r^2 + \cdots + r^n = (1 - r^{n+1})/(1-r)$ if r is not equal to 1.

Proof : If $n = 0$. $r^0 = \frac{1 - r^{0+1}}{1-r} = 1 = 1$ holds for $n = 0$

Assume $n = k \geq 0$ and the formula holds for k.

Let $n = k+1$

$$1 + r + r^2 + \cdots + r^k + r^{k+1} = \frac{1 - r^{k+1}}{1-r} + r^{k+1} \quad \text{by induction}$$

$$= \frac{1 - \cancel{r^{k+1}} + \cancel{r^{k+1}} - r^{k+2}}{1-r}$$

$$= \frac{1 - r^{k+2}}{1-r} \quad \text{holds for } n = k+1.$$

*5) Using the properties of "<" prove that the interval $(a, b) = \{ x \text{ in } \mathbb{R} \mid |x - c| < r \}$ where $c = (a+b)/2$ and $r = (b-a)/2$.

Proof: $|x - c| < r \Leftrightarrow x - c < r$ and $-x + c < r$

$$x - \frac{a+b}{2} < \frac{b-a}{2} \qquad -x + \frac{a+b}{2} < \frac{b-a}{2}$$

$$x - \frac{a+b}{2} + \frac{a+b}{2} < \frac{b-a}{2} + \frac{a+b}{2} \qquad -x + \frac{a+b}{2} - \frac{a+b}{2} < \frac{b-a}{2} - \frac{a+b}{2}$$

$$x < \frac{b-a+a+b}{2} \qquad -x < \frac{b-a-a-b}{2}$$

$$x < \frac{2b}{2} \qquad -x < -a$$

$$x > a$$

$$x < b \quad \text{and}$$

$\therefore\ a < x < b \Leftrightarrow |x - c| < r$