Solutions – HW #2
Math 554

#1 F is an ordered field & $a \in F$ satisfies $|a| < d$ $(\forall d > 0)$, then $a = 0$.

proof Using the trichotomy property we have either
(i) $a = 0$ [done]
(ii) $a > 0 \Rightarrow |a| = a > 0$
(iii) $a < 0 \Rightarrow |a| = -a > 0$

so in the remaining cases $|a| > 0$. Choose $d = \frac{1}{2}|a|$, then $d > 0$ & by our hypothesis $|a| < d$. But this just says $|a| < \frac{1}{2}|a|$. Since $|a| > 0$, then multiply the last inequality by $|a|^{-1}$ to get $1 < \frac{1}{2}$ ⨳. Only possible case is that $a = 0$. ∎

Page 30

#4(a) $\left(223/71 > 22/7\right) \Leftrightarrow \left((223)7 > (22)(71)\right)$, [multiplying <'s by positive #'s]

$\Leftrightarrow (1561 > 1562)$

Since this last statement is false, then so is the first.

#5a $\left(3(x+2) < (x+5)\right) \Leftrightarrow (3x+6 < x+5)$, [add $-x-6$]

$\Leftrightarrow (2x < -1)$, [multiply by $\frac{1}{2} > 0$]

$\Leftrightarrow \boxed{x < -1/2}$

#5b $(x^2 - 5x - 6 \geq 0) \Leftrightarrow (x-6)(x+1) \geq 0$

$\Leftrightarrow$ either $((x-6) \geq 0$ & $(x+1) \geq 0)$
or $((x-6) \leq 0$ & $(x+1) \leq 0)$

$\Leftrightarrow$ either $(x \geq 6$ and $x \geq -1)$ or $(x \leq 6$ & $x \leq -1)$

$\Leftrightarrow$ $\boxed{\text{either } (x \geq 6) \text{ or } (x \leq -1)}$

#5c $\frac{2}{x} > x-1$ Determine all such x.

Soln Need to clear the denominator, i.e. multiply by x.

By trichotomy there are 3 cases

Either **Case 1** $x>0$

then $(2 > x(x-1)) \iff ((x-2)(x+1) < 0)$

$\iff ((x<2) \,\&\, (x>-1)) \text{ or } \underbrace{(x>2 \,\&\, x<-1)}_{\text{empty set}}$

$\iff (-1<x<2)$.

But also $x>0$ in the case under consideration so

$0<x<2$

or **case 2** $x=0$

no solutions here since $\frac{2}{x}$ is undefined.

or **case 3** $x<0$

then $(2 < x^2-x) \iff (0<(x-2)(x+1))$

$\iff \underbrace{((2<x) \,\&\, (-1<x))}_{\text{empty set}} \text{ or } ((x<2) \,\&\, (x<-1))$

But $x<0$ in the case under consideration, so

$(2 < x^2-x) \iff (x<-1)$

Solution set $\boxed{\{x \mid 0<x<2 \text{ or } x<-1\}}$ i.e. $(0,2) \cup (-\infty,-1)$

#6 If $\begin{matrix} a<x<b \\ a<y<b \end{matrix}$, then $\begin{matrix} a<x<b \\ -b<-y<-a \end{matrix}$, so upon subtracting we obtain $-(b-a)<x-y<(b-a)$. But we have proved that this is equivalent to $|x-y|<b-a$. This is the same as $|y-x|<b-a$.

\# 10(a) $\boxed{\text{glb}}$ 0 is a lower bound for the set since $n \in \mathbb{N} \Rightarrow \frac{1}{n} > 0$. Suppose 0 is not the glb, then $\exists\ \varepsilon > 0$ which is a lower bound for the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$. But the Archemedian Principle states that for any $\varepsilon > 0$ $\exists\ \frac{1}{n} < \varepsilon$ for some $n \in \mathbb{N}$. If $\varepsilon > 0$, then ε cannot be a lower bound. Hence 0 is the greatest lower bound for the set.

\# 10(b) $A = \{\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \ldots, \frac{\frac{1}{2}3^n - 1}{3^n}, \ldots$

Note the general term is:

$$a_n = \frac{\frac{1}{2}3^n - 1}{3^n} = \frac{1}{2} - \left(\tfrac{1}{3}\right)^n$$

This shows $a_n < a_{n+1}$ & $a_n \le \frac{1}{2}$ all $n \in \mathbb{N}$.

Hence $\frac{1}{2}$ is an upper bound. Letting $r = \frac{1}{3}$ & that we know $0 < r^n < \cdots < r^2 < r < 1$ with $\text{glb}\{r^n \mid n \in \mathbb{Z}\} = 0$. Using this we can see that $\frac{1}{2} - \varepsilon$ is not an upper bound for A since $\exists\ n \in \mathbb{N} \ni \frac{1}{2} - \varepsilon < \frac{1}{2} - \left(\frac{1}{3}\right)^n$ (i.e. $\left(\frac{1}{3}\right)^n < \varepsilon$).

$\Rightarrow \boxed{\text{lub } A = \frac{1}{2}}$

Since $a_n < a_{n+1}$, it follows that $\boxed{\text{glb} A = \frac{1}{3}}$

Grad. Student \#11 If $a > 1$, then $a - 1 > 0$. By the Archemedian Principle $\exists\ n_0 \in \mathbb{N} \ni a - 1 > \frac{1}{n_0}$. Hence for this n_0, $a > 1 + \frac{1}{n_0}$.

But by our inequality

$$1 + n\alpha < (1+\alpha)^n \quad , n \geq 2$$

we obtain

$$2 = 1 + n_0\tfrac{1}{n_0} < \left(1 + \tfrac{1}{n_0}\right)^{n_0} < a^{n_0}$$

But then $4 < a^{n_0} a^{n_0} = a^{2n_0}$ & so by induction

$$2^j < a^{j \cdot n_0}$$

$\therefore \{a, a^2, a^3, \ldots, a^{j \cdot n_0}, \ldots\}$ cannot be bounded ∎