MATH 532, SPRING 2008

Handout # 1 , February 5

Axioms for Finite Affine Geometry

Axiom 1. There exists at least 4 points, so that when taken any 3 at a time are not co-linear.
Axiom 2. There exists at least one line incident to exactly n points.
Axiom 3. Given two (distinct) points, there is a unique line incident to both of them.
Axiom 4. Given a line l and a point P not incident to l, there is exactly one line incident to P which does not intersect l.

For the remainder of this section, all our results are stated for an affine geometry of order n.

Notation: We denote by Q, R, S, T the four points guaranteed by Axiom 1, and by P_j, $(1 \leq j \leq n)$ the points on l_0 guaranteed by Axiom 2.

Lemma 1. There exists a point P_0 which lies on exactly $n+1$ lines.

Proof. By Axiom 1 we have four points and we know at least two of them cannot belong to l_0. Denote by P_0 to be either one of these two points (so P_0 is one of Q, R, S, T). By Axiom 2, there are n distinct points on l_0, which we denote by P_j, $1 \leq j \leq n$. By Axiom 3 there are lines l_j which are incident to P_0 and, respectively, each of the points P_j. These n lines must be distinct, otherwise the points P_j would not be distinct. By Axiom 4, there is one and only one line through P_0 which does not intersect l_0, call this line l_{n+1}. If there are any other lines through P_0, then they would have to intersect l_0 at a new point P_{n+1}, but l_0 has exactly n points, so there can be no other lines through P_0. $\square$

Corollary 1. Each line l contains at least one point.

Proof. Consider the point P_0 from the previous lemma. If P_0 belongs to l, then we are done. If it does not, then by the previous lemma, there exists $n+1$ lines through P_0. But then, Axiom # 4 implies that one and only one of these lines misses l, so n of them must intersect l. Hence, l has at least n points. $\square$

Lemma 2. Fix J, $1 \leq J \leq n$, and denote by P be any of the points Q, R, S, or T. If $P \neq P_j$, then there exists a line m which 'misses' both P and P_J.

Proof. Without loss of generality, we may suppose that P from the previous lemma is the point Q. Consider the line m_1 which passes through the pair of points R, S and the line m_2 which passes through the pair of points R, T.
Neither of these lines contains $P := Q$ or then we would have a selection of 3 of these 4 special points which would be co-linear. If both of the lines contained the point P_J, then they must in fact be the same line (Axiom 3), since they would have two distinct common points, R and P_J. $\square$

Theorem 1. Each point P lies on exactly $n+1$ lines.

Proof. Fix the arbitrary point P. If P does not lie on l_0, then the proof in the first lemma shows how to construct the unique $n+1$ lines through P (i.e., one and only one line through P and the points of $P_j \in l_0$, and exactly one line "parallel" to l_0 through P).

If, instead, $P \in l_0$, then pick one of the special 4 points, $P_0 := Q$ say, which is not on l_0. Also, let J be the index so that $P = P_J$ since P belongs to l_0. By the previous lemma, there is a line m which misses both P_J and P_0. Since there are exactly $n+1$ lines going through P_0 and exactly one of these is parallel to m, the rest must intersect m. Hence there are exactly n points which lie on m (otherwise there would be another line through P_0).
Now m has the property that it misses P_J, so the number of lines through P_J (with the exception of the one line parallel to m) are in one to one correspondence with the points on m as was guaranteed by Axiom 4. Denote these points as Q_j $(1 \leq j \leq n)$
Those n points are the only points on m. Changing our view to that of P_J, and recalling that m misses P_J, we see that there are n lines from P_J and to the points Q_j. Again by Axiom 4, there is one additional line through P_J which 'misses' m, i.e. is parallel to m. Hence there are a total of exactly $n+1$ lines through P_J, which is our point P. □

Lemma 3. Each line l has exactly $n-1$ lines which do not intersect l.

Proof. (Exercise # 14 Homework. Solution on Tuesday) □

Theorem 2. Each line l has exactly n points.

Proof. From Axiom 1 (i.e. 4 special points), we know that there is some point, P_0 which is not on l. There are $n+1$ lines through P_0, and exactly one of those does not meet l. All the remaining n lines do meet l and result in n distinct points of intersection. These are the only points on l, since the points on l are in one to one correspondence with the lines through P_0 which meet l. □

Theorem 3. There are exactly n^2 points and $n\,(n+1)$ lines.

Proof. Fix any line l. By Lemma 3, there are $n-1$ lines which do not intersect l. Moreover, these lines do not intersect each other (why?). By Theorem 2, each line has n points. We know each point belongs to one of these lines by Axiom 4. So these n lines (which includes l) partitions the points into n disjoint sets of points, with n points in each of the sets. This shows that there are exactly n^2 points.

To count the number of lines, we use l_0. There are exactly $n-1$ lines which do not intersect l_0. This statement is true by Lemma 3. If we include l_0 itself, then there are n lines in the same 'direction' as l_0. All other lines must intersect l_0. These lines can be separated into disjoint sets, indexed by their point of intersection on l_0. At each of these n points (i.e. each of the P_j), there are exactly n lines, not counting l_0, which pass through P_j. Hence there are $((n-1)+1) + n \cdot n = n^2 + n$ lines.
□