Homework Solutions # 4

1.2.10

$\ln(2) = \ln\left(e \cdot \frac{2}{e}\right) = \ln e + \ln \frac{2}{e} = 1 + \ln \frac{2}{e}$.

By Taylor expansion $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + E_n(x)$,
where error term is

$$E_n(x) = \frac{(-1)^n}{n+1}\,(1+\xi)^{-n-1}\,x^{n+1} \quad \text{by series (10).}$$

$\ln\frac{2}{e} = \ln\left(1 + \left(\frac{2}{e} - 1\right)\right)$, so we apply (10) with $x = \frac{2}{e} - 1$
≈ -0.26.

The speed of convergence is determined by $|E_n| =$

$$= \frac{x^{n+1}}{(n+1)(1+\xi)^{n+1}} \leq \frac{x^{n+1}}{n+1}.$$

For $\ln 2$ represented as $\ln(1+x)$, $x=1$, rate is $\frac{1}{n+1}$.
For $\ln 2$ represented as $1 + \ln(1+x)$, $x = \frac{2}{e} - 1$, rate is $\frac{0.26^{n+1}}{n+1}$.

1.2.14

$f(x) = x^3 - 2x^2 + 4x - 1$ around $x = 2$.
Well, polynomial of 3rd degree should be represented exactly: $f(2) = 7$.

$f'(x) = 3x^2 - 4x + 4$	evaluate at $x=2$ $\quad f'(2) = 8$
$f''(x) = 6x - 4$	$f''(2) = 8$
$f'''(x) = 6$	$f'''(2) = 6$
$f^{IV} = f^{V} = f^{(n)} = 0 \quad \forall n \geq 4$	

Then $f(2+h) = 7 + 8 \cdot h + \frac{8}{2!}h^2 + \frac{6}{3!}h^3 + 0 + 0 \dots$.

1.2.26.

$f(x) = 3x^2 - 7 + \cos x =$

$= 3x^2 - 7 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) =$

$= -6 + \underbrace{\left(3 - \frac{1}{2}\right)}_{= \frac{5}{2}} x^2 + \frac{x^4}{4!} + \dots$.

1.2.39

$f(x) = \sin(x)$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos(x)$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$ $\qquad$ evaluate at $x = \frac{\pi}{4}$ $\qquad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f^{IV}(x) = \sin x$ $\qquad f^{IV}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

etc

$45.0005° = 45° + 0.0005° = \frac{\pi}{4} + \frac{\pi}{180} \cdot \frac{5}{10000} = \frac{\pi}{4} + \frac{\pi}{36} \cdot 10^{-4}$.

Then $\sin\left(\frac{\pi}{4} + h\right) =$

$= \underbrace{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot h}\ - \underbrace{\frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot h^2 - \frac{1}{\sqrt{2}} \cdot \frac{1}{3!} h^3}\ + \underbrace{\dots} = \{\text{group}\}$

$= \frac{1}{\sqrt{2}} \left[(1 + h) - \left(\frac{h^2}{2} + \frac{h^3}{6}\right) + \left(\frac{h^4}{24} + \frac{h^5}{5!}\right) + \dots \right] =$

$= \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \left[\frac{h^{2k}}{(2k)!} + \frac{h^{2k+1}}{(2k+1)!} \right]$. This is an alternating series - all a_k are positive and tend to 0 monotonically.

1.2.39...

So, plug $h = \frac{\pi}{36} \cdot 10^{-4} \approx 0.8727 \cdot 10^{-5}$. (use 9 digits)

$\sin\left(\frac{\pi}{4}+h\right) \sim \frac{1}{\sqrt{2}} \cdot \left(1+0.8727 \cdot 10^{-5}\right) -$

$- \frac{1}{\sqrt{2}} \cdot \left(3.807\,728 \cdot 10^{-11}\right)$. We can stop adding terms, since the absolute value of the last term is already less than necessary precision.

Thus, $\sin\left(\frac{\pi}{4}+h\right) \approx 0.70711295$