Math/Stat 511 Solns HW #13

Section 5.1, page 233

#11 $f(x,y) = \frac{x+y}{32}$ $x=1,2$; $y=1,2,3,4$ is the joint pmf

(a) $f_1(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \boxed{\frac{4x+10}{32}, \quad x=1,2}$

(b) $f_2(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \boxed{\frac{3+2y}{32} \quad y=1,2,3,4}$

(c) $P(X>Y) = P(X=2, Y=1) = \frac{2+1}{32} = \boxed{\frac{3}{32}}$

↑ only case when $x=1,2$; $y=1,2,3,4$

(d) $\{(X,Y) \mid Y=2X\} = \{(1,2),(2,4)\}$

so $P(Y=2X) = f(1,2)+f(2,4)$

$= \boxed{\frac{9}{32}}$

(e) $P(X+Y=3) = f(1,2)+f(2,1)$

$= \boxed{\frac{6}{32}}$

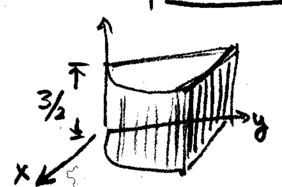
(f) $P(X \le 3-Y) = f(1,1)+f(1,2)+f(2,1)$

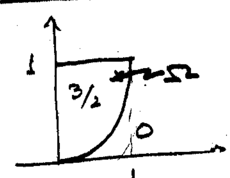
$= \frac{8}{32}$

(g) $f(1,2) = \frac{3}{32}$, $f_1(1) = \frac{14}{32}$ & $f_2(2) = \frac{7}{32}$

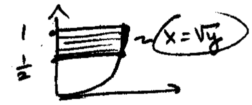
$\Rightarrow f(1,2) \neq f_1(1)f_2(2)$ ∴ $\boxed{X \,\&\, Y \text{ are dependent}}$

#8 $f(x,y) = \begin{cases} 3/2 & 0 \le x \le 1, \; x^2 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$



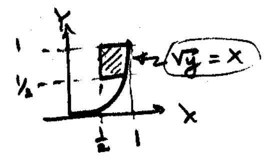


(b) $P(\frac{1}{2} \le Y \le 1) = \iint_{\Omega \cap \{\frac{1}{2} \le Y \le 1\}} \frac{3}{2} \, dx\,dy = \frac{3}{2}\int_{\frac{1}{2}}^{1}\int_{0}^{\sqrt{y}} dx\,dy$



$= \frac{3}{2}\int_{1/2}^{1}\sqrt{y}\,dy = y^{3/2}\Big|_{1/2}^{1} = \boxed{1 - \frac{1}{\sqrt{8}}}$

(c) $P(\frac{1}{2} \le X \le 1, \frac{1}{2} \le Y \le 1) = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{\sqrt{y}} \frac{3}{2} \, dx\,dy$



$= \frac{3}{2}\int_{\frac{1}{2}}^{1}(\sqrt{y} - \frac{1}{2})\,dy = \frac{3}{2}\left(\frac{2}{3}y^{3/2} - \frac{1}{2}y\right)\Big|_{\frac{1}{2}}^{1} = \boxed{\frac{5}{8} - \frac{1}{\sqrt{8}}}$

(e) $f_1(x) = \int f(x,y)\,dy = \int_{x^2}^{1} \frac{3}{2}\,dy = \frac{3}{2}(1-x^2), \quad 0 \le x \le 1$

$f_2(y) = \int f(x,y)\,dx = \int_{0}^{\sqrt{y}} \frac{3}{2}\,dy = \frac{3}{2}\sqrt{y}, \quad 0 \le y \le 1$

$f(x,y) \neq f_1(x)f_2(y)$

X & Y are dependent

Section 5.2, p.242

x	x^2	$f_1(x)$
1	1	14/32
2	4	18/32

y	y^2	$f_2(y)$
1	1	5/32
2	4	7/32
3	9	9/32
4	16	11/32

#1

$$\mu_X = \sum_{x=1}^{2} x f_1(x) = 1\cdot\frac{14}{32} + 2\cdot\frac{18}{32} = \boxed{\frac{25}{16}}$$

$$\mu_Y = \sum_{y=1}^{4} y f_2(y) = 1\cdot\frac{5}{32} + 2\cdot\frac{7}{32} + 3\cdot\frac{9}{32} + 4\cdot\frac{11}{32} = \boxed{\frac{45}{16}}$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = 1\cdot\frac{14}{32} + 4\frac{18}{32} - \left(\frac{25}{16}\right)^2 = \frac{86}{32} - \left(\frac{25}{16}\right)^2 = \boxed{\frac{63}{256}}$$

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = 1\cdot\frac{5}{32} + 4\cdot\frac{7}{32} + 9\cdot\frac{9}{32} + 16\cdot\frac{11}{32} - \left(\frac{45}{16}\right)^2 = \boxed{\frac{295}{256}}$$

$$\mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y = \sum_{x=1}^{2}\sum_{y=1}^{4} xy\,\frac{x+y}{32} - \left(\frac{25}{16}\right)\left(\frac{45}{16}\right)$$

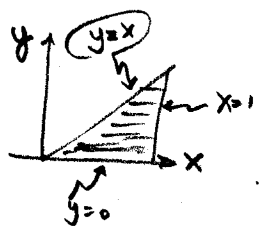
$$= 1\cdot\left(1\cdot\frac{2}{32} + 2\cdot\frac{3}{32} + 3\cdot\frac{4}{32} + 4\cdot\frac{5}{32}\right) + 2\cdot\left(1\cdot\frac{3}{32} + 2\cdot\frac{4}{32} + 3\cdot\frac{5}{32} + 4\cdot\frac{6}{32}\right) - \left(\frac{25}{16}\right)\left(\frac{45}{16}\right)$$

$$= \boxed{-\frac{5}{256}}$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{-5/256}{\sqrt{\frac{63}{256}}\sqrt{\frac{295}{256}}} \approx \boxed{-.03667}$$

X & Y are dependent from earlier problem.

#10



(a) Marginals: $f_1(x) = \int f(x,y)\,dy = \int_{0\le y\le x} f(x,y)\,dy = \int_0^x 2\,dy =$

$$\boxed{f_1(x) = 2x,\ 0\le x\le 1}$$

$$f_2(y) = \int f(x,y)\,dx = \int_{y\le x\le 1} 2\,dx = \boxed{2(1-y),\ 0\le y\le 1}$$

(b) $\mu_X = \int_0^1 x f_1(x)\,dx = \int_0^1 x\, 2x\,dx = \frac{2}{3}x^3\Big|_0^1 = \boxed{\frac{2}{3}}$

$\mu_Y = \int_0^1 y f_2(y)\,dy = \int_0^1 y\, 2(1-y)\,dy = 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \boxed{\frac{1}{3}}$

$\sigma_X^2 = E[X^2] - \mu_X^2 = \int_0^1 x^2 f_1(x)\,dx - \left(\frac{2}{3}\right)^2 = \int_0^1 x^2\, 2x\,dx - \frac{4}{9} = \frac{2}{4}x^4\Big|_0^1 - \frac{4}{9}$

$\boxed{\sigma_X^2 = 1/18}$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = \int_0^1 y^2\, 2(1-y)\,dy - \left(\frac{1}{3}\right)^2 = 2\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 - \frac{1}{9} =$

$\boxed{\sigma_Y^2 = \frac{1}{18}}$

$\mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y \qquad E[XY] = \int_0^1\int_0^x xy\cdot 2\,dy\,dx = \frac{1}{4}$

$\mathrm{Cov}(X,Y) = \frac{1}{4} - \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \boxed{\frac{1}{36}}$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\frac{1}{36}}{\frac{1}{18}} = \boxed{\frac{1}{2}}$$