Differential Equations
(Math 242.01)
Test #3 – Nov. 22, 2002

1	(10 *pts*)
2	(9 *pts*)
3	(9 *pts*)
4	(9 *pts*)
5	(9 *pts*)
6	(9 *pts*)
7	(9 *pts*)
8	(9 *pts*)
9	(9 *pts*)
10	(9 *pts*)
11	(9 *pts*)

Name: Answer

Directions: Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room. You must show intermediate work for partial credit. Integral tables are allowed.

1. Use variation of parameters to find a particular solution for the differential equation $y'' - 4y = 2e^{3t}$.

$y_p(t) = -y_1(t)\int \frac{y_2(t)}{W(t)} f(t)\,dt + y_2(t)\int \frac{y_1(t)}{W(t)} f(t)\,dt$, where $y_1(t) = e^{2t}$, $y_2(t) = e^{-2t}$, & $f(t) = 2e^{3t}$. Therefore $W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{vmatrix} = -4$

$$\therefore\ y_p(t) = -e^{2t}\cdot\int \frac{e^{-2t}}{-4} 2e^{3t}\,dt + e^{-2t}\int \frac{e^{2t}}{-4} 2e^{3t}\,dt$$

$$= -e^{2t}\left(-\tfrac{1}{2}\right)\int e^{t}\,dt + e^{-2t}\left(-\tfrac{1}{2}\right)\int e^{5t}\,dt$$

$$= \tfrac{1}{2}e^{3t} - \tfrac{1}{10}e^{-2t}e^{5t} = \boxed{\tfrac{2}{5}e^{3t}}$$

2. Use the definition of the Laplace transform to compute $L[f](s)$ where

$$f(t) = \begin{cases} 0, & 0 < t \le 1; \\ t, & 1 < t \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_1^\infty t e^{-st}\,dt = \frac{1}{-s} t e^{-st}\Big|_{t=1}^{t=\infty} + \frac{1}{s}\int_1^\infty e^{-st}\,dt$$

$$= \frac{e^{-s}}{s} + \frac{1}{s^2}e^{-s} = \boxed{\frac{s+1}{s^2}e^{-s}}$$

Compute the Laplace transform of each of the following:

3. $e^{3t}\sin(2t)$ Use 1^{st} translation thm

$$\mathcal{L}\{e^{3t}\sin(2t)\}(s) = \boxed{\frac{2}{(s-3)^2+2^2}}$$

4. $te^{3t}\sin(2t)$ Use differentiation property

$$\mathcal{L}\{te^{3t}\sin(2t)\}(s) = -\mathcal{L}\{e^{3t}\sin(2t)\}'(s)$$

$$= -\left(\frac{2}{s^2-6s+13}\right)'$$

$$= \boxed{\frac{4s-12}{(s^2-6s+13)^2}}$$

5. $(t^2+1)(2-e^t)$

$$\mathcal{L}\{(t^2+1)(2-e^t)\}(s) = \mathcal{L}\{2t^2 - t^2e^t + 2 - e^t\}(s)$$

$$= \boxed{\frac{4}{s^3} - \frac{2}{(s-1)^3} + \frac{2}{s} - \frac{1}{s-1}}$$

using 1^{st} trans. thm

6. $f(t) := \begin{cases} 0, & 0 < t < 2; \\ 3t+5, & 2 \le t < \infty. \end{cases}$ by using the unit step (Heaviside) function.

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\mathcal{U}(t-2)(3t+5)\}(s) = \mathcal{L}\{\mathcal{U}(t-2)(3(t-2)+11)\}(s)$$

$$= 3\,\mathcal{L}\{\mathcal{U}(t-2)\cdot(t-2)\}(s) + 11\cdot\mathcal{L}\{\mathcal{U}(t-2)\cdot 1\}(s)$$

$$= 3\cdot e^{-2s}\frac{1}{s^2} + 11\cdot e^{-2s}\frac{1}{s} = \boxed{\frac{3+11s}{s^2}e^{-2s}}$$

using 2^{nd} translation thm

Compute the inverse Laplace transform of each of the following:

7. $\dfrac{2s+1}{s^2(s-1)}$ $= \dfrac{3}{s-1} + \dfrac{-3s}{s^2} + \dfrac{-1}{s^2} = 3\dfrac{1}{s-1} - 3\dfrac{1}{s} - 1\cdot\dfrac{1}{s^2}$

partial fractions

$\xrightarrow{\mathcal{L}^{-1}} 3e^t - 3\cdot 1 - 1\cdot t = \boxed{3e^t - 3 - t}$

1st trans. thm

8. $\dfrac{5s+2}{s^2-2s+5}$ $= \dfrac{5(s-1)+7}{(s-1)^2+2^2} = 5\dfrac{s-1}{(s-1)^2+2^2} + \dfrac{7}{2}\dfrac{2}{(s-1)^2+2^2}$

$\xrightarrow{\mathcal{L}^{-1}} \boxed{5\cdot e^t\cos 2t + \frac{7}{2}e^t\sin 2t}$

using the 1st translation thm

9. $\dfrac{\cosh(s)}{s+1}$ $= \frac{1}{2}e^{s}\dfrac{1}{s-(-1)} + \frac{1}{2}e^{-s}\dfrac{1}{s-(-1)} \xrightarrow{\mathcal{L}^{-1}} \boxed{\frac{1}{2}\mathcal{U}(t+1)e^{-(t+1)} + \frac{1}{2}\mathcal{U}(t-1)e^{-(t-1)}}$

using 2nd translation thm

Solve each of the following equations for $y(t)$:

10. $y'(t) - \int_0^t y(s)\,ds = 1, \quad y(0) = -1.$ Apply the Laplace transform to get

$$\left(s \cdot Y(s) - (-1)\right) - \frac{Y(s)}{s} = \frac{1}{s}.$$ Solve for $Y(s)$:

$$Y(s) = \frac{1-s}{s^2-1} = -\frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \boxed{-e^{-t}}$$

11. $y * y = t$ Apply the Laplace transform & the convolution theorem to get

$$Y(s) \cdot Y(s) = \frac{1}{s^2}$$

or

$$Y(s) = \pm \frac{1}{s}$$

$\Rightarrow$ $\boxed{\text{either } y(t) = +1, \text{ or } y(t) = -1}$