

§3.5 Cyclic Groups

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MATH 546/701I

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- $\mathbf{Z}_{mn} \cong \mathbf{Z}_m \times \mathbf{Z}_n$ if $\gcd(m, n) = 1$.

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The subgroups of \mathbb{Z}

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The subgroups of \mathbf{Z}_n

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The subgroups of \mathbf{Z}_n cont.

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(a) $\langle [k]_n \rangle = \langle [d]_n \rangle \stackrel{!}{=} \langle [1]_n \rangle \Leftrightarrow d = \gcd(k, n) \stackrel{!}{=} 1$ (Why?) (see Eg. 15 in §3.2)

(b) Every subgroup of \mathbf{Z}_n is cyclic & $\langle [k]_n \rangle = \langle [d]_n \rangle$ with $d = \gcd(k, n)$. ✓

(c) \Leftarrow : $d_2|d_1 \Rightarrow d_1 = d_2q \Rightarrow [d_1]_n \in \langle [d_2]_n \rangle \Rightarrow \langle [d_1]_n \rangle \subseteq \langle [d_2]_n \rangle$ (Why?)

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The subgroups of \mathbf{Z}_n cont.

Corollary 5 (In additive notation: $G = \mathbf{Z}_n$)

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 $k = 2, 4, 8, 10, 14, 16$.

Question 2

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- $\langle [k]_n \rangle = \mathbf{Z}_n \Leftrightarrow \gcd(k, n) = 1$, i.e., $[k]_n \in \mathbf{Z}_n^\times$.
- Every subgroup of \mathbf{Z}_n is of the form $\langle [d]_n \rangle$ where $d|n$.
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So, the **subgroups of \mathbf{Z}_n** are in one to one correspondence with the **divisors of n** .

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The divisors of 18 are: 1, 2, 3, 6, 9, 18. So the subgroups of \mathbf{Z}_{18} are:

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| $[d]_{18}$ | $\langle [d]_{18} \rangle$ | $ \langle [d]_{18} \rangle $ |
|------------|---|------------------------------|
| [1] | \mathbf{Z}_{18} | 18 |
| [2] | $\{[0], [2], [4], [6], [8], [10], [12], [14], [16]\}$ | 9 |
| [3] | $\{[0], [3], [6], [9], [12], [15]\}$ | 6 |
| [6] | $\{[0], [6], [12]\}$ | 3 |
| [9] | $\{[0], [9]\}$ | 2 |
| [18] | $\{[0]\}$ | 1 |

Subgroup diagram

Notation: $m\mathbf{Z}_n = \langle [m]_n \rangle$ consisting of all multiples of $[m]_n$ in \mathbf{Z}_n .

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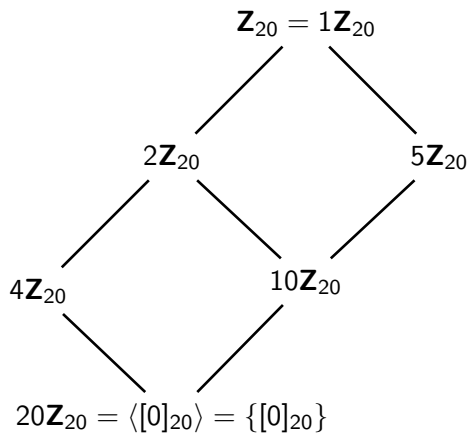
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That is, **smaller divisors of n** correspond to **larger subgroups**.

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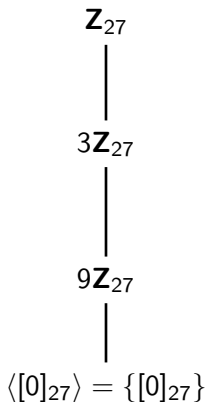
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- The order of an element is the least common multiple of the orders of each component.

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By Theorem 11, we have $\mathbf{Z}_{10} \cong \mathbf{Z}_2 \times \mathbf{Z}_5$ and $\mathbf{Z}_{20} \cong \mathbf{Z}_4 \times \mathbf{Z}_5$. By Lemma 13, we then have $\mathbf{Z}_4 \times \mathbf{Z}_{10} \cong \mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_5$ and $\mathbf{Z}_2 \times \mathbf{Z}_{20} \cong \mathbf{Z}_2 \times \mathbf{Z}_4 \times \mathbf{Z}_5$. Finally, it is easy to see that $\mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_5 \cong \mathbf{Z}_2 \times \mathbf{Z}_4 \times \mathbf{Z}_5$. (What is ϕ ?)

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Similarly, $\mathbf{Z}_4 \times \mathbf{Z}_{15} \cong \mathbf{Z}_4 \times \mathbf{Z}_3 \times \mathbf{Z}_5$ and $\mathbf{Z}_6 \times \mathbf{Z}_{10} \cong \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_2 \times \mathbf{Z}_5$.

Examples

Lemma 13

If $G_1 \cong H_1$ and $G_2 \cong H_2$, then $G_1 \times G_2 \cong H_1 \times H_2$. (What is ϕ ?)

Let $\theta_1 : G_1 \rightarrow H_1$ and $\theta_2 : G_2 \rightarrow H_2$. Define $\phi : G_1 \times G_2 \rightarrow H_1 \times H_2$ by

$$\phi((x_1, x_2)) = (\theta_1(x_1), \theta_2(x_2)), \text{ for all } (x_1, x_2) \in G_1 \times G_2.$$

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Exponent of group G

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Characterize cyclic groups among all finite abelian groups

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Using this concept of the “**exponent**” of a group, we just characterize cyclic groups among all finite abelian groups in [Proposition 2 \(b\)](#).

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Theorem 19 (The Primitive Root Theorem)

\mathbf{Z}_n^\times is cyclic if and only if $n = 1, 2, 4, p^k$ or $2p^k$ where p is any odd prime.

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- (a) Assume $o(a)$ has the largest order. To show $o(a)$ = the exponent of G .
Proof by contradiction: Let $b \in G$ and suppose that $o(b)$ is not a divisor of $o(a)$. Then there exists a prime p with $o(a) = p^\alpha n$ and $o(b) = p^\beta m$, where $(p, n) = (p, m) = 1$ and $\beta > \alpha \geq 0$. (Why?)
Then $o(a^{p^\alpha}) = n$ and $o(b^m) = p^\beta$, so these orders are relatively prime. It follows from Lemma 18 that $o(a^{p^\alpha} b^m) = np^\beta > o(a)$: contradiction.
Thus $o(b) | o(a)$ for all $b \in G$, and $o(a)$ is therefore the exponent of G .
- (b) Part (b) follows immediately from part (a). (Why?)
 G is cyclic if and only if there exists an element of order $|G|$.

Question 3

When is \mathbf{Z}_n^\times cyclic?

Theorem 19 (The Primitive Root Theorem)

\mathbf{Z}_n^\times is cyclic if and only if $n = 1, 2, 4, p^k$ or $2p^k$ where p is any odd prime.

However, we *won't prove or use* this theorem in this course.

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In conclusion, there is no element of order 8, thus the group is not cyclic.

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