Homework 5

Due: June 1st (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded..

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(1) Show that the multiplicative group $\mathbf{Z}_7^\times$ is isomorphic to the additive group $\mathbf{Z}_6$.

(2) Show that the multiplicative group $\mathbf{Z}_8^\times$ is isomorphic to the group $\mathbf{Z}_2 \times \mathbf{Z}_2$.

(3) Show that $\mathbf{Z}_5^\times$ is not isomorphic to $\mathbf{Z}_8^\times$ by showing that the first group has an element of order 4 but the second group does not.

(4) Let $(G, \cdot)$ be a group. Define a new binary operation $*$ on G by the formula
$$a * b = b \cdot a, \text{ for all } a, b \in G.$$
Show that the group $(G, *)$[1] is isomorphic to the group $(G, \cdot)$.

(5) Find two abelian groups of order 8 that are not isomorphic.

(6) Let G be any group, and let a be a fixed element of G. Define a function
$$\phi_a : G \to G \text{ by } \phi_a(x) = axa^{-1}, \text{ for all } x \in G.$$
Show that ϕ_a is an isomorphism.

(7) Let G be any group. Define $\phi : G \to G$ by $\phi(x) = x^{-1}$, for all $x \in G$.
   (a) Prove that ϕ is one-to-one and onto.
   (b) Prove that ϕ is an isomorphism if and only if G is abelian.

(8) Define $*$ on $\mathbf{R}$ by $a * b = a + b - 1$, for all $a, b \in \mathbf{R}$. Show that the group $(\mathbf{R}, *)$[2] is isomorphic to the group $(\mathbf{R}, +)$.

(9) Let $G = \mathbf{R} - \{-1\}$. Define $*$ on G by $a * b = a + b + ab$. Show that the group $(G, *)$[3] is isomorphic to the multiplicative group $\mathbf{R}^\times$.

(10) Let $G = \{x \in \mathbf{R} \mid x > 1\}$. Define $*$ on G by $a * b = ab - a - b + 2$, for all $a, b \in G$. Define $\phi : G \to \mathbf{R}^+$ by $\phi(x) = x - 1$, for all $x \in G$.
   (a) Show that $(G, *)$ is a group.
   (b) Show that ϕ is an isomorphism.

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[1] In Homework 2 (3), we have shown that $(G, *)$ is a group.
[2] In Homework 2 (7), we have shown that $(\mathbf{R}, *)$ is a group.
[3] In Homework 2 (8), we have shown that $(G, *)$ is a group.