Homework 3

Due: May 22nd (Friday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) In $\mathrm{GL}_2(\mathbf{R})$, find the order of each of the following elements.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(2) For each of the following groups, find all cyclic subgroups of the group.

(a) $\mathbf{Z}_8$

(b) $\mathbf{Z}_{12}^{\times}$

(3) Find the cyclic subgroup of S_6 generated by the element $(123)(456)$.

(4) Let $G = \mathrm{GL}_3(\mathbf{R})$. Show that

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \right\}$$

is a subgroup of G.

(5) Let S be a set, and let a be a fixed element of S. Show that

$$\{\sigma \in \mathrm{Sym}(S) \mid \sigma(a) = a\}$$

is a subgroup of $\mathrm{Sym}(S)$.

(6) Prove that any cyclic group is abelian.

(7) Prove that the intersection of any collection of subgroups of a group is again a subgroup.

(8) Let G be a group, and let $a \in G$. The set

$$C(a) = \{x \in G \mid xa = ax\}$$

of all elements of G that commute with a is called the **centralizer** of a.

(a) Show that $C(a)$ is a subgroup of G.

(b) Show that $\langle a \rangle \subseteq C(a)$.

(c) Computer $C(a)$ if $G = S_3$ and $a = (123)$.

(d) Computer $C(a)$ if $G = S_3$ and $a = (12)$.

(9) Let G be a group. The set
$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$$
of all elements that commute with every other element of G is called the **center** of G.

(a) Show that $Z(G)$ is a subgroup of G.

(b) Show that $Z(G) = \cap_{a \in G} C(a)$.

(c) Computer the center of S_3.

(10) Show that if a group G has a unique element a of order 2, then $a \in Z(G)$.

(11) Let G be a group with $a, b \in G$.

(a) Show that $o(a^{-1}) = o(a)$.

(b) Show that $o(ab) = o(ba)$.

(c) Show that $o(aba^{-1}) = o(b)$.

(12) Let G be a group with $a, b \in G$. Assume that $o(a)$ and $o(b)$ are finite and relatively prime, and that $ab = ba$. Show that $o(ab) = o(a)o(b)$.