Strategy for Integration

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MATH 2924-915

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Outline for section 1

1. Before solving the integration
   - Integration formulas
   - Simplify the integrand if possible
2. Five methods for solving integration
   - $\mathcal{U}$-substitution
   - Integration by parts
   - Rational functions
   - Trigonometric substitution
   - Rationalizing substitution
3. Improper Integrals

Integration formulas

Table of Integration Formulas More formulas can be found elsewhere...

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin(\frac{x}{a}) + C$$

$\vdots$

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Example

$$\int \frac{x}{\sqrt{x}(1+x)}\,dx = \int \frac{\sqrt{x}}{1+x}\,dx = \cdots$$

Outline for section 2

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   - Integration formulas
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2. Five methods for solving integration
   - $\mathcal{U}$-substitution
   - Integration by parts
   - Rational functions
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Rational functions $\frac{P(x)}{Q(x)}$ (**Partial Fractions**)

1. If $\deg(P(x)) \geq \deg(Q(x))$, do the long division calculation first:

Example

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

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   - Linear factors (of the form $ax + b$);
   - Irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

Example

$$Q(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$$

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$$\frac{A}{(ax+b)^i}, \qquad \frac{Ax+B}{(ax^2+bx+c)^j}, \qquad (i, j \geq 1)$$

Rational functions $\frac{P(x)}{Q(x)}$ (**Partial Fractions**), II

4. The denominator $Q(x)$ is a product of distinct linear factors:

Example

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} =$$

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$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Trigonometric substitution

1. $\sqrt{a^2 - x^2}$, $x = a\sin\theta$ and use Identity $1 - \sin^2\theta = \cos^2\theta$.

Example

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2. $\sqrt{a^2 + x^2}$, $\quad x = a\tan\theta$ and use Identity $1 + \tan^2\theta = \sec^2\theta$.

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3. $\sqrt{x^2 - a^2}$, $x = a\sec\theta$ and use Identity $\sec^2\theta - 1 = \tan^2\theta$.

Example

$$\int \frac{1}{\sqrt{x^2 - 4}}\,dx, \quad x = 2\sec\theta, \quad dx = 2\sec\theta\tan\theta\,d\theta$$

Rationalizing substitution

Some nonrational functions (eg. $\sqrt{x+1}$, $\sqrt[n]{ax+b}$) can be changed into rational functions by means of appropriate substitutions.

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$$\int \sqrt{x}e^{\sqrt{x}}\,dx, \quad u=\sqrt{x}, \quad x=u^2, \quad dx=2u\,du$$

Outline for section 3

Improper Integrals

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Good Luck !