My primary research interest lies at the intersection of number theory and representation theory. I have also developed a strong interest in the Langlands program, particularly the relation between automorphic forms and automorphic representation theory. First, I will give a quick summary of my publications and preprints, which can be classified into two types. Then more detailed explanations will follow in Sect. 1. Lastly, I will describe several ongoing and future projects in Sect. 2.

I. Local and global representation theory, classical modular forms

i) In my Ph.D. thesis [21] (or [22]), I calculate the dimensions of the spaces of invariant vectors under certain congruence subgroups for the local representations of GSp(4); see Theorem 1.1. This kind of dimensional data can be used to determine the dimension formulas of corresponding classical Siegel cusp forms, which is one of several ongoing projects; see Sect. 2.1.

ii) In [14] (joint with Manami Roy and Ralf Schmidt), we find the number of cuspidal automorphic representations of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ under certain conditions. See Theorem 1.4 for the asymptotic version of results. As an immediate consequence, we obtain the limit multiplicity formula (8), which can be generalized to the vector-valued case and a finite number of ramified places by using the automorphic Plancherel density theorem.

iii) In [10] (joint with Chiranjit Ray and Manami Roy), we give some congruences involving dimensions of spaces of Siegel cusp forms of degree 2 and the class number of $\mathbb{Q}(\sqrt{-p})$; for example see Theorem 1.6 for the case $p \equiv 1 \pmod 4$. As a consequence, we obtain a result related to the 4-core partition function; see Corollary 1.7.

iv) In [13] (joint with Manami Roy and Ralf Schmidt), we carry out "Hecke summation" for the classical Eisenstein series E_k in an adelic setting. More precisely, we determine the automorphic representations generated by the E_k, particularly for $k = 2$, where the representation is neither a pure tensor nor has finite length; see Theorem 1.8. We also consider Eisenstein series of weight 2 with level, and Eisenstein series with character; see Theorems 1.9 and 1.10.

II. Number theory in function fields, abstract analytic number theory

i) In [4] (joint with Lian Duan and Biao Wang), we show an analogue of Kural, McDonald and Sah's result on Alladi's formula, which displays a relationship between the Möbius function and the density of primes in arithmetic progressions, for global function fields; see Theorem 1.11.

ii) Furthermore, in [3] (joint with Lian Duan and Ning Ma), we prove that a general version of Alladi's formula with Dirichlet convolution holds for arithmetical semigroups satisfying Axiom A or Axiom $A^{\#}$; see Theorems 1.14 and 1.16. As a result, we apply the main results to certain semigroups coming from algebraic number theory, arithmetical geometry and graph theory.

1 Main results

1.1 Klingen $\mathfrak{p}^2$ vectors for GSp(4)

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. The algebraic group GSp(4) is defined as

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4)\colon\ {}^t gJg = \lambda(g)J,\ \lambda(g) \in \mathrm{GL}(1)\}, \quad J = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}. \tag{1}$$

The homomorphism $\lambda\colon \mathrm{GSp}(4) \longrightarrow \mathrm{GL}(1)$ is called the multiplier homomorphism. Its kernel is the symplectic group $\mathrm{Sp}(4)$. There are three different conjugacy classes of parabolic subgroups of $\mathrm{GSp}(4)$, represented by the Borel subgroup B, Siegel parabolic subgroup P and Klingen parabolic subgroup Q. Moreover, we define the following congruence subgroups:

$$\text{The Klingen congruence subgroup } \mathrm{Kl}(\mathfrak{p}^2) := \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{o} \end{bmatrix}, \tag{2}$$

$$\text{The \textit{middle group} } \mathrm{M}(\mathfrak{p}^2) := \{g \in \mathrm{GSp}(4,F)\colon\ \det(g) \in \mathfrak{o}^\times\} \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{o} \end{bmatrix}. \tag{3}$$

The irreducible admissible representations of $\mathrm{GSp}(4,F)$ come in two classes. The first class consists of all those representations that can be obtained as subquotients of parabolically induced representations from one of the parabolic subgroups B, P or Q. These representations have already been classified and described in [15] by Sally and Tadić. Furthermore, Roberts and Schmidt reproduce the list; see [11, Sect. 2.2]. The second class consists of all the other representations which are called supercuspidal. Here, we also list all these representations in the Table 1 further below.

Theorem 1.1. *The dimensions of the spaces of* $\mathrm{M}(\mathfrak{p}^2)$*- and* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors for all irreducible, admissible representations of* $\mathrm{GSp}(4,F)$ *with trivial central character, where* F *is a* $\mathfrak{p}$*-adic field, have been determined.*

In particular, Table 1 gives the dimensional data for the special case of $F = \mathbb{Q}_2$ and so $\mathfrak{p}^2 = 4\mathbb{Z}_2$; see [21, 22] for the general results. Moreover, Table 1 will be used to determine the dimension formulas of corresponding Siegel cusp forms, which is one of several ongoing projects; see Sect. 2.1.

Case I: Iwahori-spherical representations An admissible representation (π, V) of $\mathrm{GSp}(4,F)$ is called *Iwahori-spherical* if it has non-zero I-fixed vectors with the Iwahori subgroup I defined as

$$I := \mathrm{GSp}(4,\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{bmatrix}. \tag{4}$$

Note that the irreducible Iwahori-spherical representations are exactly the constituents of the Borel-induced representations with an unramified character of $B(F)$. To find the desired dimensional data in this case, the main methods are *double coset decompositions* and *intertwining operators.*

Case II: Non Iwahori-spherical representations The following lemma shows that only the depth zero representations can admit non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors.

Lemma 1.2. *Let* (π, V) *be a smooth representation of* $\mathrm{GSp}(4,F)$*. Suppose the space of* $\mathrm{Kl}(\mathfrak{p}^2)$*-invariant vectors is non-zero. Then* π *is a depth zero representation, i.e., the space of invariant vectors under the principal congruence subgroup* $\Gamma(\mathfrak{p})$ *is non-zero.*

To obtain the desired dimensional data, we use a variety of methods: *double coset decompositions*, *intertwining operators*, *P_3-theory* ([11, Sect. 2.5]) and *hyperspecial parahoric restriction* ([12]).

1.2 On counting cuspidal automorphic representations for GSp(4) (with Manami Roy and Ralf Schmidt)

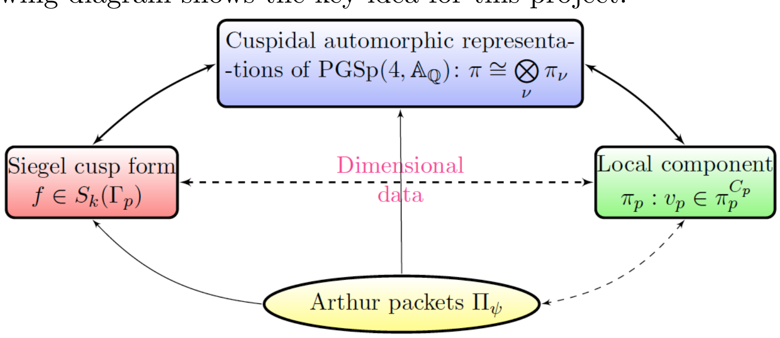
Definition 1.3. Let $G = \mathrm{GSp}(4)$. Let k be a positive integer, and let p be a prime. Let $S_k(p,\Omega)$ be the set of cuspidal automorphic representations $\pi \cong \otimes_{\nu\leq\infty}\pi_\nu$ of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character

Table 1: Dimensions of $\mathrm{M}(4\mathbb{Z}_2)$ and $\mathrm{Kl}(4\mathbb{Z}_2)$-invariant vectors for $\mathrm{GSp}(4,\mathbb{Q}_2)$

		constituent of	representation	$\mathrm{M}(4\mathbb{Z}_2)$	$\mathrm{Kl}(4\mathbb{Z}_2)$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)		8	11
II	a	$\nu^{1/2}\chi\times\nu^{-1/2}\chi\rtimes\sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	5	7
	b	$(\chi^2\neq\nu^{\pm1},\chi\neq\nu^{\pm3/2})$	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	3	4
III	a	$\chi\times\nu\rtimes\nu^{-1/2}\sigma$	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	3	5
	b	$(\chi\notin\{1,\nu^{\pm2}\})$	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	5	6
IV	a	$\nu^2\times\nu\rtimes\nu^{-3/2}\sigma$	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	1	2
	b		$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	2	3
	c		$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	4	5
	d		$\sigma 1_{\mathrm{GSp}(4)}$	1	1
V	a	$\nu\xi\times\xi\rtimes\nu^{-1/2}\sigma$	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	3	5
	b	$(\xi^2=1,\ \xi\neq1)$	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	2	2
	c		$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	2	2
	d		$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	1	2
VI	a	$\nu\times 1_{F^\times}\rtimes\nu^{-1/2}\sigma$	$\tau(S,\nu^{-1/2}\sigma)$	3	5
	b		$\tau(T,\nu^{-1/2}\sigma)$	0	0
	c		$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	2	2
	d		$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	3	4
VII		$\chi\rtimes\pi$ (irreducible)		0	2
VIII	a	$1_{F^\times}\rtimes\pi$	$\tau(S,\pi)$	0	2
	b		$\tau(T,\pi)$	0	0
IX	a	$\nu\xi\rtimes\nu^{-1/2}\pi$	$\delta(\nu\xi,\nu^{-1/2}\pi)$	0	1
	b	$(\xi\neq1,\ \xi\pi=\pi)$	$L(\nu\xi,\nu^{-1/2}\pi)$	0	1
X		$\pi\rtimes\sigma$ (irreducible)		2	3
XI	a	$\nu^{1/2}\pi\rtimes\nu^{-1/2}\sigma$	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	1	2
	b	$(\omega_\pi=1)$	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	1	1
s.c.		depth zero generic supercuspidal		0	1
		other s.c.		0	0

such that π_∞ is the lowest weight module with minimal K-type (k,k), and π_p is an Iwahori-spherical representation of $G(\mathbb{Q}_p)$ of type Ω and unramified otherwise. Here, the representation type Ω is one of the types listed in Table 1: I, IIa, IIb, ..., VId.

By general principles $S_k(p,\Omega)$ is finite. Let $s_k(p,\Omega):=\#S_k(p,\Omega)$ be its cardinality. Roughly speaking, the following diagram shows the key idea for this project:



More precisely, we consider $\pi \cong \otimes\pi_\nu \in S_k(p,\Omega)$ as in Definition 1.3. In particular, π_ν is unramified for every $\nu \neq p,\infty$, which implies that $v_\nu \in V_\nu$ is the distinguished unramified vector. As for $\nu = p$, let C_p be one of the compact open subgroups of $G(\mathbb{Q}_p)$ stabilizing a non-zero vector v_p in V_p, and let Γ_p be the corresponding congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$; for example see (10). Then we have the following equation, which is the basis for our determination of $s_k(p,\Omega)$.

$$\dim_{\mathbb{C}} S_k(\Gamma_p) = \sum_{\Omega}\sum_{\pi\in S_k(p,\Omega)} \dim \pi_p^{C_p} = \sum_{\Omega} s_k(p,\Omega)\,\dim \pi_p^{C_p}. \tag{5}$$

Here the quantities $\dim \pi_p^{C_p}$, which is the same across all Iwahori-spherical representations of type Ω, are given explicitly in [16, Table 3]. We also note that the dimension formulas $\dim_{\mathbb{C}} S_k(\Gamma_p)$ for the spaces of Siegel modular forms of degree 2 are known; for example see [6, 5, 7, 18, 19].

To calculate the quantities $s_k(p,\Omega)$, we need another important input, i.e., Arthur packets. Note that the split orthogonal group $\mathrm{SO}(5)$ is isomorphic to $\mathrm{PGSp}(4)$ as algebraic groups. By [2] the Arthur packets fall into six classes: the finite type **(F)**, the general type **(G)**, the Yoshida type **(Y)**, and the types **(P)**, **(B)** and **(Q)** consisting mostly of CAP representations (cuspidal associated to parabolics). Cuspidal representations cannot be finite-dimensional, so we will ignore the type **(F)**.

Let $S_k^{(*)}(p,\Omega)$ be the set of those $\pi \in S_k(p,\Omega)$ that lie in an Arthur packet of type $(*)$. Let $s_k^{(*)}(p,\Omega) = \#S_k^{(*)}(p,\Omega)$ be its cardinality. First, we prove that $s_k^{(\mathbf{Q})}(p,\Omega) = s_k^{(\mathbf{B})}(p,\Omega) = 0$ for all k and all Ω; see [14, Cor. 2.3]. It follows that for all $k \geq 1$, we have

$$s_k(p,\Omega) = s_k^{(\mathbf{G})}(p,\Omega) + s_k^{(\mathbf{P})}(p,\Omega) + s_k^{(\mathbf{Y})}(p,\Omega), \tag{6}$$

where $s_k^{(\mathbf{Y})}(p,\Omega) = 0$ unless $\Omega = \mathrm{VIb}$. Then we find explicit formulas for $s_k(p,\Omega)$; see [14, Thms. 3.1, 3.3-3.8]. In the following, we just give the asymptotic version of results for simplicity.

Theorem 1.4. *For $k \geq 2$ and $\Omega \in \{\mathrm{IIb}, \mathrm{Vb}, \mathrm{VIb}^{(\mathbf{P})}, \mathrm{VIb}^{(\mathbf{Y})}, \mathrm{VIc}\}$, $s_k(p,\Omega) = O_p(k)$. For $k \geq 3$ and $\Omega \in \{\mathrm{I}, \mathrm{IIa}, \mathrm{IIIa}+\mathrm{VIa/b}, \mathrm{IVa}, \mathrm{Va}\}$, $s_k(p,\Omega) = \frac{a_\Omega}{2^7 3^3 5}(k-2)(k-1)\big(2k-3\big) + O_p(k^2)$, where*

Ω	I	IIa	IIIa + VIa/b	IVa	Va
a_Ω	1	p^2-1	$\frac{1}{2}(p-1)(p^2+p+2)$	$(p-1)(p^3-1)$	$\frac{1}{2}p(p-1)^2$

(7)

It turns out that a_Ω in (7) equals to the total Plancherel measure m_Ω of the tempered Iwahori-spherical representations of $G(\mathbb{Q}_p)$ with trivial central character of type Ω; see [14, Thm. 4.2]. As an immediate consequence, we obtain the following limit multiplicity formula

$$\lim_{k\to\infty} \frac{s_k(p,\Omega)}{2^{-7}3^{-3}5^{-1}(k-2)(k-1)(2k-3)} = m_\Omega. \tag{8}$$

Moreover, using the automorphic Plancherel density theorem from [17], we generalize (8) to the vector-valued case and a finite number of ramified places; see [14, Thm. 4.5]. Note that (8) also holds for local representations of other types Ω; for example see the ongoing project in Sect. 2.1.

1.3 Congruences for dimensions of spaces of Siegel cusp forms and 4-core partitions (with Chiranjit Ray and Manami Roy)

Definition 1.5. A Siegel modular form $f : \mathcal{H}_2 \to \mathbb{C}$ of degree 2 and weight k with respect to a congruence subgroup Γ_N of $\mathrm{Sp}(4,\mathbb{Q})$ is a holomorphic function satisfying the following property

$$(f|_k g)(Z) = \det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1}) = f(Z) \quad \text{for } g = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] \in \Gamma_N. \tag{9}$$

We call a Siegel modular form f a *cusp form* if $\lim\limits_{\lambda\to\infty} (f|_k g)\left(\left[\begin{smallmatrix} i\lambda & \\ & \tau\end{smallmatrix}\right]\right) = 0$ for all $g \in \mathrm{Sp}(4,\mathbb{Q}), \tau \in \mathcal{H}$.

For our purpose, consider the full modular group $\mathrm{Sp}(4,\mathbb{Z})$ and the following congruence subgroups:

$$\text{The paramodular group } \mathrm{K}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Q}),$$
$$\text{The Klingen congruence subgroup } \Gamma_0'(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Z}), \tag{10}$$
$$\text{The Siegel congruence subgroup } \Gamma_0(N) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4,\mathbb{Z}).$$

Let $S_k^{\mathrm{new}}(\Gamma_0^{(1)}(p))$ be the new subspace of weight k elliptic cusp forms on the congruence subgroup $\Gamma_0^{(1)}(p)$ of $\mathrm{SL}(2,\mathbb{Z})$. Using the relationship between Siegel cusp forms of degree 2 and cuspidal automorphic representations of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ (see (5)), we derive the following congruences results.

Theorem 1.6. *Let $h(\Delta_p)$ be the class number of $\mathbb{Q}(\sqrt{-p})$. For $k \geq 3$ and $p \equiv 1 \pmod 4$, we have*

$$\begin{aligned} &4\Big(\dim_\mathbb{C} S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\dim_\mathbb{C} S_2^{\mathrm{new}}(\Gamma_0^{(1)}(p)) - (-1)^{k-1}\dim_\mathbb{C} S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\Big) \\ &+8\Big(\dim_\mathbb{C} S_k(\mathrm{K}(p)) - \dim_\mathbb{C} S_k(\Gamma_0(p))\Big) \equiv (-1)^{k-1}h(\Delta_p)^2 - \begin{cases} 4h(\Delta_p) & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \pmod{16}, \\ &4\Big(\dim_\mathbb{C} S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\dim_\mathbb{C} S_2^{\mathrm{new}}(\Gamma_0^{(1)}(p)) + \dim_\mathbb{C} S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(p))\Big) \\ &\qquad\qquad +8\Big(\dim_\mathbb{C} S_k(\mathrm{K}(p)) + \dim_\mathbb{C} S_k(\Gamma_0(p))\Big) \equiv (-1)^{k-1}h(\Delta_p)^2 \pmod{16}. \end{aligned}$$

The similar results can be obtained for other primes p as well; see [10, Thms. 3.1-3.2] for more details. Furthermore, by using Theorem 1.6, we also obtain some congruences between the 4-core partition function $c_4(n)$ and dimensions of spaces of Siegel cusp forms of degree 2; see [10, Cor. 4.1] for explicit details. As a consequence, we prove the following result.

Corollary 1.7. *Suppose $8n+5$ is a prime number. Then $c_4(n) \equiv \dim_\mathbb{C} S_{4k}^{\mathrm{new}}(\Gamma_0^{(1)}(8n+5)) \pmod 2$.*

1.4 Classical and adelic Eisenstein series (with Manami Roy and Ralf Schmidt)

Let $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_{\mathrm{fin}}$ be the global Hecke algebra, where $\mathcal{H}_{\mathrm{fin}} = \bigotimes_{p<\infty} \mathcal{H}_p$. It acts on the space $\mathcal{A}$ of automorphic forms on $\mathrm{GL}(2,\mathbb{A})$. Any irreducible subquotient is called an automorphic representation. The $\mathrm{GL}(2,\mathbb{A})$-representation generated by an automorphic form Φ is $\mathcal{H}\Phi$. Let E_k be the classical Eisenstein series, and let Φ_k be the automorphic form corresponding to E_k. Let $V_{s,p} = |\cdot|_p^s \times |\cdot|_p^{-s}$ for $s \in \mathbb{C}$. For $s = 1/2$ and $s = -1/2$ we have the exact sequences

$$0 \longrightarrow \mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_p)} \longrightarrow V_{1/2,p} \longrightarrow 1_{\mathrm{GL}(2,\mathbb{Q}_p)} \longrightarrow 0, \tag{11}$$
$$0 \longrightarrow 1_{\mathrm{GL}(2,\mathbb{Q}_p)} \longrightarrow V_{-1/2,p} \longrightarrow \mathrm{St}_{\mathrm{GL}(2,\mathbb{Q}_p)} \longrightarrow 0. \tag{12}$$

Theorem 1.8. *If $k \geq 4$ is an even integer, then the $\mathcal{H}$-module π generated by Φ_k is irreducible. We have $\pi \cong \bigotimes \pi_v$, with $\pi_\infty = \mathcal{D}_{k-1}^{\mathrm{hol}}$, the discrete series representation of lowest weight k, and $\pi_p = V_{(k-1)/2,p}$, an irreducible principal series representation, for all $p < \infty$. If $k = 2$, then the global representation $\mathcal{H}\Phi_2$ generated by Φ_2 admits a filtration $0 \subset \mathbb{C} \subset \mathcal{H}\Phi_2$, where $\mathbb{C}$ is the space of constant automorphic forms, and $\mathcal{H}\Phi_2/\mathbb{C} \cong \mathcal{D}_1^{\mathrm{hol}} \otimes \bigotimes_{p<\infty} V_{1/2,p}$ as $\mathcal{H}$-modules.*

Let ξ be a primitive Dirichlet character with conductor $u > 1$ and χ the corresponding character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Let $E_{k,\xi}$ be the classical Eisenstein series with character ξ, and let $\Phi_{k,\chi}$ be the automorphic form corresponding to $E_{k,\xi}$. Let $V_{s,\chi_p} = \chi_p|\cdot|_p^s \times \chi_p^{-1}|\cdot|_p^{-s}$ for $s \in \mathbb{C}$. It is reducible if and only if $\chi_p = \mu_p|\cdot|_p^{-s\pm1/2}$ with a quadratic character μ_p. In this case, we have $V_{s,\chi_p} = \mu_p V_{\pm1/2,p}$, and it has the analogous exact sequences as in (11) and (12), i.e., twisted by μ_p.

Theorem 1.9. *If $k \geq 4$ is an even integer, then the $\mathcal{H}$-module π generated by $\Phi_{k,\chi}$ is irreducible. We have $\pi \cong \bigotimes \pi_v$, with $\pi_\infty = \mathcal{D}_{k-1}^{\rm hol}$ and $\pi_p = V_{(k-1)/2,\chi_p}$ for all $p < \infty$. If $k = 2$, then the global representation $\mathcal{H}\Phi_{2,\chi}$ generated by $\Phi_{2,\chi}$ is*

$$\mathcal{H}\Phi_{2,\chi} \cong \mathcal{D}_1^{\rm hol} \otimes \bigotimes_{\substack{p|u\\ \chi_p^2\neq1}} V_{1/2,\chi_p} \otimes \bigotimes_{\substack{p|u\\ \chi_p^2=1}} \chi_p{\rm St}_{{\rm GL}(2,\mathbb{Q}_p)} \otimes \bigotimes_{p\nmid u} V_{1/2,\chi_p} \quad \textit{as } \mathcal{H}\textit{-modules.} \tag{13}$$

We consider the "natural" Eisenstein series $E_{2,N}$ with level N ([13, Thm. 5.5]) since it has a more natural Fourier expansion than the classical Eisenstein series $\tilde{E}_{2,N}$ with level N. Moreover, the global representations generated by the automorphic forms corresponding to the $\tilde{E}_{2,N}$ are in general not tensor products. We clarify the relationship between these two types of functions; see [13, Prop. 5.6]. Let $E_{2,N,\xi}$ be the "natural" Eisenstein series with character ξ; see [13, Thm. 6.6].

Theorem 1.10. *For a square-free positive integer $N > 1$ let $\Phi_{2,N}$ be the automorphic form corresponding to $E_{2,N}$. Then $\mathcal{H}\Phi_{2,N} \cong \mathcal{D}_1^{\rm hol} \otimes \Big(\bigotimes_{p|N} {\rm St}_{{\rm GL}(2,\mathbb{Q}_p)}\Big) \otimes \Big(\bigotimes_{p\nmid N} V_{1/2,p}\Big)$. Suppose that $\chi_p^2 = 1$ for $p \mid N$, let $\Phi_{2,N,\chi}$ be the automorphic form corresponding to $E_{2,N,\xi}$. Then*

$$\mathcal{H}\Phi_{2,N,\chi} \cong \mathcal{D}_1^{\rm hol} \otimes \bigotimes_{\substack{p|u\\ \chi_p^2\neq1}} V_{1/2,\chi_p} \otimes \bigotimes_{\substack{p|u\\ \chi_p^2=1}} \chi_p{\rm St}_{{\rm GL}(2,\mathbb{Q}_p)} \otimes \bigotimes_{p|N} \chi_p{\rm St}_{{\rm GL}(2,\mathbb{Q}_p)} \otimes \bigotimes_{p\nmid uN} V_{1/2,\chi_p}. \tag{14}$$

1.5 Analogues of Alladi's formula over global function fields (with Lian Duan and Biao Wang)

The prime number theorem (PNT) says that the number of primes up to x is asymptotic to $x/\log x$. As is well known, the PNT is closely related to the Möbius function since it is equivalent to $\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0$. Alladi [1] rewrote it as $-\sum_{n=2}^\infty \frac{\mu(n)}{n} = 1$, and showed that if $(\ell, k) = 1$, then

$$-\sum_{n\geq2,\, p(n)\equiv\ell(\mathrm{mod}k)} \frac{\mu(n)}{n} = \frac{1}{\varphi(k)}, \tag{15}$$

where $p(n)$ is the smallest prime divisor of n, and φ is Euler's totient function. Motivated by the result of Kural, McDonald and Sah [9], we show the analogous result over global function fields.

Let $\mathbb{F}_q$ be a finite field with q elements. Let K be a global function field with constant field $\mathbb{F}_q$. The *norm* of a *prime divisor* P in K is defined by $\|P\| = q^{\deg P}$. A *divisor* D is a finite formal sum of prime divisors, i.e., $D = \sum_P a_P \cdot P$ such that $a_P \in \mathbb{Z}$ for every P and $a_P = 0$ for all but finitely many P. A divisor D is called *effective* if $a_P \geq 0$ for all P. Let $\mathcal{D}$ be the set consisting of all the divisors of K. The subset of all effective divisors is denoted by $\mathcal{D}^+$ and the subset consisting of all prime divisors is denoted by $\mathcal{P}$.

For any subset $S \subseteq \mathcal{P}$, we say S has a *natural density* $\delta(S) := \lim\limits_{n\to\infty} \pi_{K,S}(n)/\pi_K(n)$ provided the limit exists, where $\pi_{K,S}(n) := \#\{P \in S : \deg P = n\}$, and $\pi_K(n) := \pi_{K,\mathcal{P}}(n)$.

Let $d_-(D) := \min\{\deg P : P \mid D\}$. We say that D is *distinguishable* if $D \neq 0$ and there is a unique prime factor, say $P_{\min}(D)$, of D attaining the minimal degree $d_-(D)$. We define

$$\mathfrak{D}(K,S) := \left\{D \in \mathcal{D}^+ : D \text{ is distinguishable and } P_{\min}(D) \in S\right\}. \tag{16}$$

Theorem 1.11. *Given a global function field K, if $S \subseteq \mathcal{P}$ has a natural density $\delta(S)$, then*

$$-\lim_{n\to\infty} \sum_{\substack{1\le \deg D\le n\\ D\in\mathfrak{D}(K,S)}} \frac{\mu(D)}{\|D\|} = \delta(S). \tag{17}$$

Here, the Möbius function $\mu(D) = (-1)^k$ if D is the sum of k distinct primes and zero otherwise.

1.6 Generalizations of Alladi's formula for arithmetical semigroups (with Lian Duan and Ning Ma)

Definition 1.12. An *arithmetical semigroup* is a pair $(\mathcal{G}, \|\cdot\|)$. Here $\mathcal{G}$ is a commutative semigroup with the identity element $e_{\mathcal{G}}$, which is freely generated by a finite or countable subset $\mathcal{P}$ of *primes* of $\mathcal{G}$. In addition, there exists a discrete real-valued *norm mapping* $\|\cdot\|$ on $\mathcal{G}$ such that

(1) $\|e_{\mathcal{G}}\| = 1$, and $\|P\| > 1$ for every $P \in \mathcal{P}$.

(2) $\|gh\| = \|g\|\cdot\|h\|$ for all $g, h \in \mathcal{G}$.

(3) For each $x \in \mathbb{R}_{>0}$, the total number $\mathcal{N}(x)$ of elements $g \in \mathcal{G}$ of norm $\|g\| \le x$ is finite.

Definition 1.13. An arithmetical semigroup $\mathcal{G}$ satisfies *Axiom A* if

$$\mathcal{N}(x) = c_{\mathcal{G}}x + O(x^{\eta}) \qquad \text{as } x \to \infty \tag{18}$$

with suitable constants $c_{\mathcal{G}} > 0$ and $0 < \eta < 1$.

For any subset $S \subseteq \mathcal{P}$, we say that S has a *natural density* $\delta(S)$, if the following limit exists:

$$\delta(S) := \lim_{x\to\infty} \frac{\pi_{\mathcal{G},S}(x)}{\pi_{\mathcal{G}}(x)}, \tag{19}$$

where $\pi_{\mathcal{G},S}(x) := \#\{P \in S : \|P\| \le x\}$ and $\pi_{\mathcal{G}}(x) := \pi_{\mathcal{G},\mathcal{P}}(x)$. Let $\mathrm{N}_-(g) := \min\{\|P\| \colon P \mid g\}$. We say that g is *distinguishable* if $g \neq e_{\mathcal{G}}$ and there is a unique prime factor, say $P_{\min}(g)$, of g attaining the minimum norm $\mathrm{N}_-(g)$. Define $\mathfrak{D}(\mathcal{G},S) := \{g \in \mathcal{G} : g \text{ is distinguishable and } P_{\min}(g) \in S\} \cup \{e_{\mathcal{G}}\}$. We will also need the natural generalization of the *Möbius function* $\mu(g)$ to elements $g \in \mathcal{G}$. More precisely, $\mu(g) = (-1)^k$ if g is the product of k distinct primes and zero otherwise.

Theorem 1.14. *Given an Axiom A type arithmetical semigroup $\mathcal{G}$, assume that $S \subseteq \mathcal{P}$ has a natural density $\delta(S)$. Suppose that $a\colon \mathcal{G} \to \mathbb{C}$ is an arithmetic function supported on $\mathfrak{D}(\mathcal{G},S)$ with $a(e_{\mathcal{G}}) = 1$, and $\lim_{x\to\infty} \sum_{\substack{2\le\|g\|\le x\\ g\in\mathfrak{D}(\mathcal{G},S)}} \frac{|a(g)|}{\|g\|} \log\log\|g\| < \infty$. Let $\mu * a$ be the Dirichlet convolution. Then*

$$-\lim_{x\to\infty} \sum_{\substack{2\le\|g\|\le x\\ g\in\mathfrak{D}(\mathcal{G},S)}} \frac{\mu * a(g)}{\|g\|} = \delta(S). \tag{20}$$

In particular, if a is the identity of the convolution ring of arithmetic functions over $\mathcal{G}$, then

$$-\lim_{x\to\infty} \sum_{\substack{2\le\|g\|\le x\\ g\in\mathfrak{D}(\mathcal{G},S)}} \frac{\mu(g)}{\|g\|} = \delta(S). \tag{21}$$

Note that (20) (resp. (21)) generalizes the main result in [20] (resp. [9]).

We can also consider a *degree mapping* ∂ on $\mathcal{G}$, which is defined as $\partial(g) := \log_c \|g\|$ for some fixed constant $c > 1$. Then the properties (1)-(3) in Definition 1.12 can be translated naturally into three corresponding properties for ∂. We call $\mathcal{G}$ together with ∂ an *additive* arithmetical semigroup.

Definition 1.15. An arithmetical semigroup $\mathcal{G}$ satisfies *Axiom* $A^{\#}$ if

$$G^{\#}(n) := \left|\{g \in \mathcal{G} \colon \partial(g) = n\}\right| = c_{\mathcal{G}} q^n + O(q^{\eta n}) \qquad \text{as } n \to \infty \tag{22}$$

with suitable constants $c_{\mathcal{G}} > 0$, $q > 1$ and $0 \leq \eta < 1$.

In addition, we assume $Z_{\mathcal{G}}(-q^{-1}) \neq 0$, where $Z_{\mathcal{G}}(z) := \prod_P (1 - z^{\partial(P)})^{-1}$ is the zeta function of $\mathcal{G}$.

Similarly, we define the *natural density* $\delta(S)$ for a subset $S \subseteq \mathcal{P}$ and the set $\mathfrak{D}^{\#}(\mathcal{G}, S)$ as in the case of Axiom A above, but in terms of the degree mapping ∂. Also the general *Möbius function* is defined by $\mu(g) = (-1)^k$ if g is the sum of k distinct primes and zero otherwise. In the following theorem, we fix the norm map $\|\cdot\| : \mathcal{G} \to \mathbb{C}$ to be $\|g\| = q^{\partial(g)}$ with the base q as in Definition 1.15.

Theorem 1.16. *Given an Axiom* $A^{\#}$ *type arithmetical semigroup* $\mathcal{G}$*, assume that* $S \subseteq \mathcal{P}$ *has a natural density* $\delta(S)$*. Suppose that* $a \colon \mathcal{G} \to \mathbb{C}$ *is an arithmetic function supported on* $\mathfrak{D}^{\#}(\mathcal{G}, S)$ *with* $a(e_{\mathcal{G}}) = 1$*, and* $\lim_{n\to\infty} \sum_{\substack{1 \leq \partial(g) \leq n \\ g \in \mathfrak{D}^{\#}(\mathcal{G},S)}} \frac{|a(g)|}{\|g\|} \log\log\|g\| < \infty$*. Let* $\mu * a$ *be the Dirichlet convolution. Then*

$$-\lim_{n\to\infty} \sum_{\substack{1 \leq \partial(g) \leq n \\ g \in \mathfrak{D}^{\#}(\mathcal{G},S)}} \frac{\mu * a(g)}{\|g\|} = \delta(S). \tag{23}$$

In particular, if a *is the identity of the convolution ring of arithmetic functions over* $\mathcal{G}$*, then*

$$-\lim_{n\to\infty} \sum_{\substack{1 \leq \partial(g) \leq n \\ g \in \mathfrak{D}^{\#}(\mathcal{G},S)}} \frac{\mu(g)}{\|g\|} = \delta(S). \tag{24}$$

Note that (24) generalizes the main result in [4], i.e., Theorem 1.11. Also note that the denominator "$\|g\|$" in (20) and (23) can be generalized to "$\varphi(g)$"; see [3, Cors. 1.3, 1.6].

2 Ongoing and future projects

2.1 Siegel modular forms of degree 2 and Klingen level 4 (with Manami Roy and Ralf Schmidt)

The main goal of this project is to find the dimension formula $\dim_{\mathbb{C}} \mathcal{S}_{k,j}(\mathrm{Kl}(4))$ of Siegel modular forms of degree 2 of weight (k, j) and Klingen level 4. The idea for this ongoing project is similar to that for the previous work [14]. However, more work is required since more local representations are involved now. This project is one application of my Ph.D. thesis results [21, 22]. In particular, Table 1 will be used to determine the desired dimension formulas. Moreover, we are also able to give the limit multiplicity formula as (8) for local representations of other types Ω.

2.2 Refinement of the effective Chebotarev density theorem over function fields (with Lian Duan and Ning Ma)

In this project, we give a refined result of the effective Chebotarev density theorem for function fields, which particularly improves the result [8] by Kumar Murty and Scherk. Moreover, we can also apply our method to the higher dimensional cases. Currently, we are trying to obtain the analogous results for finite graphs and arithmetical semigroups motivated by our previous work [3].

2.3 $\mathrm{B}(\mathfrak{p}^2)$-invariant vectors for GSp(4)

This project is to obtain the dimensions of the spaces of $\mathrm{B}(\mathfrak{p}^2)$-invariant vectors for all irreducible, admissible representations of $\mathrm{GSp}(4, F)$ by using the similar technical tools in my Ph.D. thesis. It is expected that much more work is required since the Borel congruence subgroup $\mathrm{B}(\mathfrak{p}^2)$ of level $\mathfrak{p}^2$ is smaller than $\mathrm{Kl}(\mathfrak{p}^2)$. An immediate application of this kind of result is helpful to give the dimension formula of the corresponding classical Siegel cusp forms as in Sect. 2.1.

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