https://people.math.sc.edu/shaoyun/Review_241F_SYi_Fa_21.pdf

Review for Test 1 (§12.1-12.6, §13.1-13.4)

(1) The **distance formula** $|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$.

⇝ **Equation of a sphere** centered at (a,b,c) with radius r:

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \tag{1}$$

(2) The **magnitude** or **length** of $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ is $|\mathbf{v}|=\sqrt{v_1^2+v_2^2+v_3^2}$.

(3) **Addition/Difference** of vectors & **Scalar multiplication** (parallel)

⇝ Properties of Vector Operations

(4) $\mathbf{v}=\langle v_1,v_2,v_3\rangle = v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}$, where $\mathbf{i},\mathbf{j},\mathbf{k}$ are the **standard unit vectors**.

(5) For $\mathbf{v}\neq\mathbf{0}$, $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of $\mathbf{v}$.

(6) The **midpoint** between points P_1 and P_2 is $M\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2},\dfrac{z_1+z_2}{2}\right)$.

(7) The **dot product** $\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle = u_1v_1+u_2v_2+u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$.

⇝ Properties of the Dot Product

- Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u}\cdot\mathbf{v}=0$.
- The **vector projection of u onto v** is

$$\text{proj}_{\mathbf{v}}\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}=(|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|} \text{ with scalar component } \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

- **Work** $W=\mathbf{F}\cdot\mathbf{D}$ with a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D}$.

(8) The **cross product** $\mathbf{u}\times\mathbf{v}$ is the vector

$$\mathbf{u}\times\mathbf{v}=(|\mathbf{u}||\mathbf{v}|\sin\theta)\,\mathbf{n}=\det\left(\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{bmatrix}\right), \tag{2}$$

where $\mathbf{n}$ is the unit **normal vector** perpendicular to $\mathbf{u}, \mathbf{v}$ by the right-hand rule.

$\rightsquigarrow$ Properties of the Cross Product (e.g., $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$)

- ⋆ Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
- ⋆ $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
- ⋆ **Torque** $\mathbf{T} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the vector from the axis along the lever.

(9) The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right) \tag{3}$$

$\rightsquigarrow$ $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ gives the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

(10) **A vector equation for the line** L **through** $P_0(x_0, y_0, z_0)$ **parallel to** $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \qquad = \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}, \qquad -\infty < t < \infty. \tag{4}$$

$\rightsquigarrow$ **The standard parametrization of** L:

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3, \qquad -\infty < t < \infty. \tag{5}$$

(11) Distance d from S to a line through P parallel to $\mathbf{v}$:

$$d = |\overrightarrow{PS}| \sin\theta = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{6}$$

(12) $M = \left\{P \colon \overrightarrow{P_0P} \text{ is orthogonal to } \mathbf{n} := \langle A, B, C\rangle\right\} \quad \leftrightsquigarrow \quad \mathbf{n} \cdot \overrightarrow{P_0P} = 0$

$\rightsquigarrow$ **Component equation for a plane**: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$\rightsquigarrow$ **simplified**: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$.

(13) The line of intersection of two planes is parallel to $\mathbf{n}_1 \times \mathbf{n}_2$.

(14) Distance from S to a plane through P with normal $\mathbf{n}$: $d = |\overrightarrow{PS}||\cos\theta| = \left|\dfrac{\overrightarrow{PS}\cdot\mathbf{n}}{|\mathbf{n}|}\right|$

(15) **The angle between two intersecting planes** is defined to be the **acute** angle between their normal vectors.

(16) *Cylinders and Quadric Surfaces**

(17) Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function. Then

$$\lim_{t\to t_0}\mathbf{r}(t) = \left\langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\right\rangle \qquad \text{provided the limit exists.} \tag{7}$$

$\rightsquigarrow$ $\mathbf{r}(t)$ is **continuous at** $t = t_0$ in its domain if $\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{r}(t_0)$.

(18) The **derivative** $\mathbf{r}'(t) = \dfrac{d\mathbf{r}}{dt} = \langle f'(t), g'(t), h'(t)\rangle$.

- $\mathbf{r}(t)$ is **smooth** if $\mathbf{r}'(t)$ is continuous and never $\mathbf{0}$.
- $\mathbf{r}'(t) \neq \mathbf{0}$ is called the vector **tangent** to the curve at P.
- The **tangent line** to the curve at P is the line through P parallel to $\mathbf{r}'(t)$.
- $\mathbf{r}(t)$ position vec., $\mathbf{v}(t) = \mathbf{r}'(t)$ velocity vec., $\mathbf{a}(t) = \mathbf{r}''(t)$ acceleration vec.
- **Differentiation Rules:**
  - $\star$ $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$, $\quad \dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$, $\quad \dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
  - $\star$ $\dfrac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$, $\quad \dfrac{d}{dt}\big[\mathbf{u}\big(f(t)\big)\big] = f'(t)\mathbf{u}'\big(f(t)\big)$
  - $\star$ $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
  - $\star$ $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
- If $|\mathbf{r}(t)|$ is constant for all t, then $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt} = 0$. The converse is also true.

(19) The **indefinite integral** $\displaystyle\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$, where $\mathbf{R}$ is any antiderivative of $\mathbf{r}$

- The **definite integral** $\displaystyle\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$.
- **Fundamental Theorem of Calculus** $\displaystyle\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$
- Initial value problem (IVP): $\mathbf{v}(t) = \displaystyle\int \mathbf{a}(t)\,dt, \qquad \mathbf{r}(t) = \int \mathbf{v}(t)\,dt$

  $\rightsquigarrow$ Projectile Motion: Maximum height $(y'(t) = 0)$, Flight time $(y(t) = 0)$

(20) The **length** of a smooth curve $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, t \in [a, b]$, is

$$L = \int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \tag{8}$$

- **Arc Length Parameter** $s(t)$ **with Base Point** $P(t_0) = (x(t_0), y(t_0), z(t_0))$**:**

$$s(t) = \int_{t_0}^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\,d\tau = \int_{t_0}^t |\mathbf{r}'(\tau)|\,d\tau \quad \rightsquigarrow \frac{ds}{dt} = |\mathbf{r}'(t)|$$

  Solve for t in terms of s: $\rightsquigarrow$ the curve can be reparametrized $\mathbf{r}(t) = \mathbf{r}\big(t(s)\big)$.

- The **unit tangent vector** for $\mathbf{r}(t)$ is given by $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, where $\mathbf{v}(t) = \mathbf{r}'(t)$.

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}\frac{1}{ds/dt} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}$$

  $\rightsquigarrow$ $d\mathbf{r}/ds$ is the unit tangent vector in the direction of the velocity vector $\mathbf{v}$.

(21) The **curvature** $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$, where $\mathbf{T}$ is the unit tangent vector on a smooth curve.

$\rightsquigarrow$ If $\mathbf{r}$ is smooth, then $\kappa = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right|$.

(22) The **principal unit normal** vector for a smooth curve is $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds}$ for $\kappa \neq 0$.

- The vector $\dfrac{d\mathbf{T}}{ds}$ (and so $\mathbf{N}$) points toward the concave side of the curve.
- If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$.
- $\mathbf{N}$ and $\mathbf{T}$ are orthogonal from Theorem 0.6 in §13.1 since $|\mathbf{T}| = 1$.

- ○ **Vector Formula for Curvature:** $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$
  - ⋆ If $|\mathbf{r}'| \neq 0$ is constant, then $\mathbf{r}' \perp \mathbf{r}''$. $\rightsquigarrow$ $\kappa \overset{!}{=} \dfrac{|\mathbf{r}'||\mathbf{r}''||\sin 90°|}{|\mathbf{r}'|^3} = \dfrac{|\mathbf{r}''|}{|\mathbf{r}'|^2}$
  - ⋆ Radius of Osculating Circle: $R = \dfrac{1}{\kappa}$ also called the radius of curvature.

Review for Test 2 (§14.1-14.7)

(1) interior point (belongs to R); boundary point (may not belong to R);

(2) open/closed/bounded/unbounded region R

(3) level curve/surface; contour curve/surface

(4) Properties of Limits of Functions of Two Variables

(5) Three common ways to find the limit $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$:

- plug in $(x,y) = (x_0,y_0)$ directly if $f(x,y)$ is continuous at (x_0,y_0)
- simplify $f(x,y)$ by canceling zero denominator to becoming a new function, which is continuous at (x_0,y_0)
- multiply by conjugate if $f(x,y)$ involves radicals, especially somthing like $\sqrt{\quad}$

(6) Two-Path Test for Nonexistence of a Limit

(7) Know how to find the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ etc

(8) Know how to use Chain Rule properly to find the (partial) derivatives

(9) Formulas for Implicit Differentiation

(10) Know how to use Gradient ∇f to find the directional derivatives $D_{\mathbf{u}}f$

(11) Properties of Directional Derivative $D_{\mathbf{u}}f$

(12) The gradient of f is normal to the level curve through (x_0,y_0), i.e., $\nabla f \cdot \dfrac{d\mathbf{r}}{dt} = 0$

(13) Tangent Line (resp. Plane) to a Level Curve (resp. Surface); Normal line // Gradient ∇f

(14) Algebra Rules for Gradients

(15) The Chain Rule for Paths: for example, $\dfrac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ for $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$

(16) Estimating the Change in f in a Direction $\mathbf{u}$; standard linear approximation; (total) differential

(17) Second Derivative Test for Local Extreme Values

(18) Absolute maxima/minima of $f(x,y)$ on closed bounded region & Application in real life example

Review for Test 3 (§15.1-15.5, 15.7, 16.1-16.2)

(1) Double Integrals: $\iint_R f\,dx\,dy$, $\iint_R f\,dy\,dx$, $\iint_R f\,r\,dr\,d\theta$ $\rightsquigarrow$ Find limits of integration

(2) Triple Integrals: $\iiint\limits_D F\,dz\,dy\,dx$, $\iiint\limits_D F\,dz\,r\,dr\,d\theta$, $\iiint\limits_D F\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ $\rightsquigarrow$ Limits of integration

(3) Area $\leftrightsquigarrow$ $(f=1)$; Volume $\leftrightsquigarrow$ $(F=1)$; Average value of f (resp. F) over R (resp. D)

(4) Line Integral of f over C: $\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))|\mathbf{v}(t)|\,dt$

(5) Line Integral of $\mathbf{F}$ along C: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$ $\rightsquigarrow$ Work, Circulation

(6) Line Integrals with Respect to dx, dy, or dz:

$$\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b \left(Mg'(t)+Nh'(t)+Pk'(t)\right)\,dt = \int_C M\,dx+N\,dy+P\,dz$$

Review for (§16.3-16.4)

(1) Fundamental Theorem of Line Integrals: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^B \nabla f\cdot d\mathbf{r} = f(B)-f(A)$

(2) Conservative Fields are Gradient Fields: $\mathbf{F}$ is conservative $\Leftrightarrow$ $\mathbf{F}=\nabla f$ for some scalar function f.

(3) $\mathbf{F}=\nabla f$ on D $\Leftrightarrow$ $\mathbf{F}$ conservative on D $\Leftrightarrow$ $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ over any loop in D (Loop Property)

(4) Component Test for Conservative Fields: $\mathbf{F}$ is conservative $\Leftrightarrow$ $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z}=\frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z}=\frac{\partial P}{\partial y}$.

(5) A differential form is **exact** if $M\,dx+N\,dy+P\,dz = \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz = df$

(6) The differential form $M\,dx+N\,dy+P\,dz$ is exact $\Leftrightarrow$ $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z}=\frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z}=\frac{\partial P}{\partial y}$.
This is equivalent to saying that the field $\mathbf{F}=\langle M,N,P\rangle$ is conservative.

(7) Green's Theorem: $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C M\,dx+N\,dy = \iint\limits_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

$\boxed{\text{MyMathLab HW, Class notes, Quizzes (solutions in BB)}}$

Good luck with the test!