Review for Final

Shaoyun Yi

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University of South Carolina

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Function: $y = f(x)$

- Domain/Range $\leftrightsquigarrow$ Interval notation
- Increasing/Decreasing, Even/Odd $\leftrightsquigarrow$ $f(-x) = \pm f(x)$
- Piecewise, Power ($\rightsquigarrow$ Polys, Rational), Trigonometric, Exponential
- Inverse function $\begin{cases} \text{one-to-one; Solve } y = f(x) \text{ for } x \text{, then } x \leftrightarrow y \\ y = a^x \longleftrightarrow y = \log_a x \\ \sin(x), \cos(x), \ldots \longleftrightarrow \sin^{-1}(x), \cos^{-1}(x), \ldots \end{cases}$
- Composite function $(f \circ g)(x) = f\big(g(x)\big)$
- Shifting/Scaling and Reflecting a graph of a function
- Trigonometric Identities and Formulas
- Change of Base Formulas for Exponential/Logarithmic function

$$a^x = e^{x \ln a} \qquad \log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$$

Limit: $\lim_{x\to c} f(x) = L$

- Limit Laws: $f \pm g, \quad k\cdot f, \quad f\cdot g, \quad f/g, \quad f^n, \quad f^{1/n}$
- If $P(x) = a_n x^n + \cdots + a_0$, then $\lim_{x\to c} P(x) = P(c) = a_n c^n + \cdots + a_0$.
- If $P(x)$ and $Q(x)$ are polys and $Q(c) \neq 0$, then $\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$.
- Eliminate common factors from 0 denominators/Multiply by conjugate
- The Sandwich Theorem (A fact of Trig.: Range of $\sin, \cos$ is $[-1, 1]$)
- $\delta - \varepsilon$ language (prove theorems):
  - for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
  - Find algebraically a δ for a given f, L, c and $\varepsilon > 0$
    (1) Solve $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c.
    (2) Find a $\delta > 0$ so that $(c-\delta, c+\delta)$ centered at c **inside** (a, b).
- $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$
- $\lim_{x\to\infty}, \lim_{x\to-\infty}$: Divide by highest power of x/Multiply by conjugate

Continuity: $\lim_{x \to c} f(x) = f(c)$

- The function f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.
- Four types of discontinuities: removable, jump, infinite, oscillating
- **Algebraic Combinations**: $f \pm g, \quad k \cdot f, \quad f \cdot g, \quad f/g, \quad f^n, \quad f^{1/n}$
- **Polynomials** and **Rational** functions (if well-defined) are continuous
- The **inverse** function of any continuous function on I is continuous
- All **composites** of continuous functions are continuous
- If g is continuous at b and $\lim_{x \to c} f(x) = b$, then $\lim_{x \to c} g\big(f(x)\big) = g(b)$.
- Intermediate Value Theorem (IVT) for continuous functions

Derivative (R.O.C): $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$

- The formal definition: Find derivatives & Prove differentiation rules
- Differentiable functions are continuous; The converse might be false.
- **Differentiation Rules:**
  - $(c)' = 0, \qquad (x^n)' = nx^{n-1}, \qquad (a^x)' = (\ln a)a^x \overset{a=e}{\dashrightarrow} (e^x)' = e^x$
  - $(c\cdot f)' = c\cdot f', \qquad (f \pm g)' = f' \pm g'$
  - $(f\cdot g)' = f'\cdot g + f\cdot g', \qquad \left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$
  - Second- and Higher-Order Derivatives
- **Algebraic** = **Geometric:** $f'(a)$ = Slope m of Tangent line at $x = a$

Formulas for Derivatives *(PDF is in Blackboard)*

- **Constant Rule:** $(k)' = 0$
- **Power Rule:** $(x^n)' = n\,x^{n-1}$
- **Exponential Rule:** $(a^x)' = (\ln a)\,a^x$
- **Natural Exponential Rule:** $(e^x)' = e^x$
- **Logarithmic Rule:** $(\log_a x)' = \dfrac{1}{(\ln a)\,x}$
- **Natural Logarithmic Rule:** $(\ln x)' = 1/x$
- **Trig. Rule:** $(\sin x)' = \cos x,\ (\cos x)' = -\sin x,\ (\tan x)' = \sec^2 x$
- **Constant Multiple Rule:** $(c \cdot f)' = c \cdot f'$
- **Sum/Difference Rule:** $(f \pm g)' = f' \pm g'$
- **Product Rule:** $(f \cdot g)' = f' \cdot g + f \cdot g'$
- **Quotient Rule:** $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$
- **Chain Rule:** $(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$
- **Derivative Rule for Inverses:** $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$
- **Implicit/Logarithmic differentiation**

Applications of Derivatives

- Velocity, Acceleration: $v(t) = s'(t), a(t) = v'(t) = s''(t)$; Speed$=|v(t)|$
- Rates of change and Derivatives in Economics are so-called *marginals*

$\star$ **Algebraic** $=$ **Geometric:** $f'(a) =$ Slope m of Tangent line at $x = a$

The slope of the normal $*$ The slope of the tangent line $= -1$

$\star$ Derivatives of the inverse trigonometric functions $\sin^{-1}, \cos^{-1}, \tan^{-1}$

$\star$ Derivatives of positive functions involving products, quotients, powers.

It can often be found more quickly by using Logarithmic differentiation:

$$(\ln y)' = \frac{1}{y} \cdot y' \quad \Rightarrow y' = y \cdot (\ln y)'$$

$\star$ Related Rates: Review examples in §3.10. (Implicit differentiation:)

1. Differentiate both sides of the equation w.r.t. x, treating y as a function of x.

2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Mean Value Theorem & Its applications

Mean Value Theorem

Suppose that $y = f(x)$ is continuous over $[a, b]$ and differentiable on (a, b). Then there is at least one point $c \in (a, b)$ at which $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Rolle's Theorem is a special case ($f(a) = f(b)$) of Mean Value Theorem.

- Intermediate Value Theorem & Rolle's Theorem $\Rightarrow$ "*exactly one real solution*"
- If $f'(x) = 0$ at each $x \in (a, b)$, then $f(x) = C$ (a constant) for all $x \in (a, b)$.
- If $f'(x) = g'(x)$ at each $x \in (a, b)$, then $f(x) = g(x) + C$ for all $x \in (a, b)$.

$\star$ First Derivative Test: $f' > 0$ means $f \nearrow$ **v.s.** $f' < 0$ means $f \searrow$

$\star$ Second Derivative Test: $f'' > 0$ means f 🙂 **v.s.** $f'' < 0$ means f 🙁

Global/Local Extrema & Critical/Inflection points

If $f'(c)$ is zero or undefined for an interior point c, then c is a **critical point** of f.

- Global Maxima/Minima**:** Compare critical values and endpoints values

⋆ Local Maxima/Minima**:** Critical points ($f'(c) = 0$) & f' sign changes

Methods: **2nd derivative test** ($f''(c) \neq 0$); otherwise, **1st derivative test**

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

⋆ Inflection points (f changes concavity)**:** $f''(c) = 0$ & f'' sign changes

Application: Together f' and f'' tell us the shape of the function's graph.

(a, b) Identify the domain of f and any symmetries may have; then find f' and f''

(c) Find critical points and identify function's behavior at each one. [FDT, SDT]

Find where the curve is increasing and where it is decreasing. [FDT]

(d) Find the points of inflection, and determine the concavity of the curve. [SDT]

(e) Identify any asymptotes & Plot key points (intercepts, pts in **(c)**, **(d)**)

L'Hôpital's Rule & Applied Optimization

L'Hôpital's Rule for the indeterminate form $0/0$, ∞/∞

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Indeterminate Forms $\infty \cdot 0$, $\infty - \infty$

These forms can be converted to a $0/0$ or ∞/∞ form by using algebra.

Indeterminate Forms 1^∞, 0^0, ∞^0

(1) take the logarithm; **(2)** use L'Hôpital's Rule; **(3)** exponentiate the result

$$\text{If } \lim_{x \to a} \ln f(x) = L, \text{ then } \lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{\lim_{x \to a} \ln f(x)} = e^L.$$

Solving Applied Optimization Problems (Modeling and Doing math)

1). Read the problem. **2).** Draw a picture. **3).** Introduce variables.

4). Write an equation for the unknown quantity.

5). Test the critical points and endpoints in the domain of the unknown.

Indefinite and Definite Integrals & Applications

$\int f(x)\,dx = F(x) + C$, where $F(x)$ is an antiderivative of $f(x)$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \left(\frac{b-a}{n}\right)$$ & **Properties**

FTC, I & II: $F'(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$; $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

- $$\int_a^b f(x)\,dx = \begin{cases} \text{area under the curve} & \text{if } f \geq 0 \text{ on } [a,b], \\ -\text{area below the } x\text{-axis} & \text{if } f < 0 \text{ on } [a,b]. \end{cases}$$
- Average value $(f) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$
- Substitution Rule: $u = u(x)$ & $du = u'(x)\,dx$
- $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & \text{if } f \text{ is even}, \\ 0 & \text{if } f \text{ is odd}. \end{cases}$$
- Areas Between Curves: $A = \displaystyle\int_a^b [f(x) - g(x)]\,dx$

- **Constant Rule:** $(k)' = 0$
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- **Logarithmic Rule:** $(\log_a x)' = \dfrac{1}{(\ln a)\,x}$
- **Natural Logarithmic Rule:** $(\ln x)' = \dfrac{1}{x}$
- $(\sin x)' = \cos x,\ (\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x,\ (\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \tan x,\ (\csc x)' = -\csc x \cot x$
- **Constant Multiple Rule:** $(c \cdot f)' = c \cdot f'$
- **Sum/Difference Rule:** $(f \pm g)' = f' \pm g'$
- **Product Rule:** $(f \cdot g)' = f' \cdot g + f \cdot g'$
- **Quotient Rule:** $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$
- **Chain Rule:** $(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$
- **Derivative Rule for Inverses:**

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$(\arcsin x)',\ (\arccos x)',\ (\arctan x)'$

- **Implicit/Logarithmic differentiation**

- $\displaystyle\int k\,dx = kx + C$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int \cos x\,dx = \sin x + C,\ \int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C,\ \int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C,\ \int \csc x \cot x\,dx = -\csc x + C$
- $\displaystyle\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
- $\displaystyle\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- **NA**
- **NA**
- ***u*-substitution:** $du = u'(x)\,dx$
- **NA**
- **NA**

Volumes Using Cross-Sections/Cylindrical Shells

- Volumes Using Cross-Sections $V = \int_a^b A(x)\,dx$
  1. Sketch the solid and a typical cross-section.
  2. Find a formula for $A(x)$, the area of a typical cross-section.
  3. Find the limits of integration & Integrate $A(x)$ to find the volume.
- Solids of Revolution about the x-axis
  - Disk Method: $V = \int_a^b A(x)\,dx = \int_a^b \pi\,[\,R(x)\,]^2\;dx$
  - Washer Method: $V = \int_a^b A(x)\,dx = \int_a^b \pi\left([\,R(x)\,]^2 - [\,r(x)\,]^2\right)\;dx$
- Volumes Using Cylindrical Shells $V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}\,dx$

  e.g., $y = f(x)$ is revolved about the vertical line $x = L < a < b$:

$$V = \int_a^b 2\pi(x - L)f(x)\,dx$$