Final Review

Shaoyun Yi

MATH 242

University of South Carolina

Apr 23, 2020

Definitions in ODE

I. Order of an ODE

II. Checking Solutions of ODE's

III. General Solutions vs. Particular Solutions (I.V.P.)

IV. ODE's of the form $\dfrac{dy}{dx} = f(x)$

   (i) This is the first ("simplest") type of ODE
   (ii) *Application:* Free-fall problems; acceleration/velocity/position

V. Slope Fields and Solution Curves:
   Geometric way to describe the (approximate) solution

VI. **Theorem** for Existence and Uniqueness of Solutions (for I.V.P.)

   (i) $f(x, y)$ is continuous near $(a, b) \Rightarrow$ Existence on some open interval I
   (ii) Moreover, if in addition $\partial f/\partial y$ is continuous near $(a, b) \Rightarrow$ Uniqueness on some (perhaps smaller) interval J

Separable and Linear first-order ODE

I. Separable ODE's of the form $\dfrac{dy}{dx} = g(x)k(y)$

(i) Implicit, General/Particular and Singular Solutions
(ii) *Application:* Exponential Growth and Decay

II. First-Order Linear ODE's of the form $\dfrac{dy}{dx} + P(x)y = Q(x)$

(i) Finding a Multiplier (Integrating factor)

$$\rho(x) = \exp(\int P(x)\,dx)$$

(ii) Solving the ODE

1. Multiply both sides of the equation by $\rho(x)$.
2. Recognize that the LHS of the Eq. is the derivative $D_x\left[\rho(x)y(x)\right]$.
3. Integrate both sides of the equation.

(iii) **Theorem** for Existence and Uniqueness of Solutions (for I.V.P.)
(iv) *Application:* Mixing Problems

More types of first-order ODE and Methods

I. Substitution Methods for Solving ODE's

(i) The first-order ODE of the form

$$\frac{dy}{dx} = F(ax+by+c) \quad \dashrightarrow v = ax + by + c \quad \dashrightarrow \text{Separable ODE}$$

(ii) The *homogeneous*[1] first-order ODE of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \dashrightarrow v = \frac{y}{x} \quad \dashrightarrow \text{Separable ODE}$$

(iii) Bernouli Equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \dashrightarrow v = y^{1-n} \quad \dashrightarrow \text{First-order Linear ODE}$$

(iv) *Application:* Flight Trajectories

II. Exact Equations of the form

$$F_x\, dx + F_y\, dy = 0 \text{ or } M\, dx + N\, dy = 0$$

(i) **Theorem** of Criterion for Exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

(ii) Solve the ODE

[1] This is different from the homogeneous equation in Chapter 3.

Some Applications: Mathematical models and numerical methods

I. Population Models: General Equation:

$$\frac{dP}{dt} = (\beta - \alpha)P$$

- β : birth rate; α : death rate
- Logistics Equation:

$$\frac{dP}{dt} = kP(M - P), M \text{ is the } \textit{carrying capacity}$$

II. Equilibrium Solutions and Stability:
- **Phase Diagrams**
- Bifurcation Points

III. Acceleration-Velocity Models:

$$F = ma = mx''$$

IV. Numerical Approximation[2]: (Improved) Euler's Method

[2] Won't show up in the test

Linear equations of higher order: Definitions

I. Principle of Superposition (for linear Homogeneous equations)

II. Homogeneous vs. Non-homogeneous (nth-order linear equations)

III. Complimentary solution y_c. vs. Particular solution y_p

Any solution of Non-homogeneous equation: $\boxed{y = y_c + y_p}$

IV. Linearly Independent vs. Linearly Dependent (**Wronskians**)

V. **Wronskians** (for any n functions, $n \geq 2$):

- Definition
- Calculations

Homogeneous linear ODE's with Constant Coefficients

I. Characteristic equation

II. Distinct Real Roots:

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

III. Repeated Real Roots r with Multiplicity $k \geq 2$:

$$(c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{rx}$$

IV. Unrepeated Pair of Complex Roots $a \pm bi$:

$$e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

V. Repeated Pair of Complex Roots $a \pm bi$ with Multiplicity $k \geq 2$:

$$e^{ax} \cdot \left(\sum_{p=0}^{k-1} x^p (c_p \cos bx + d_p \sin bx) \right)$$

VI. General Solutions & Solutions to I.V.P.'s

Non-Homogeneous linear ODE's with Constant Coefficients

Recall that

Any solution of Non-homogeneous equation: $\boxed{y = y_c + y_p}$

Note that y_c can be solved by the methods in previous slide.

To find y_p :

I. **Method of Undetermined Coefficients**

   (i) **Rule 1:**
   - Without difficulty (i.e. Linearly Independent already)

   (ii) **Rule 2:**
   - With difficulty (i.e. multiply by x^s to obtain Linearly Independent)

II. Method of Variation of Parameters

Applications of second-order linear equation

Springs and Damping

$$mx'' + cx' + kx = F(t)$$

I. §**3.4:** Free$\dashrightarrow$ ($F_E = F(t) = 0 \leftrightarrow$ Homogeneous equation)

- Undamped ($c = 0$): Simple harmonic motion
- Damped ($c \neq 0$):

  1. Underdamped,
  2. Critically Damped,
  3. Overdamped.

II. §**3.6:** Forced$\dashrightarrow$ ($F_E = F(t) \neq 0 \leftrightarrow$ Non-Homogeneous equation)

- Undamped ($c = 0$)

Laplace Transforms and Inverse Transforms

I. Definition of the Laplace Transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

II. The Inverse Transform: $f(t) = \mathcal{L}^{-1}\{F(s)\}$

III. Some General Properties of the Laplace Transforms
   (i) **Linearity**
   (ii) **Existence** (*Functions of Exponential Order*)
   (iii) **Uniqueness**

IV. Laplace Transforms of Some Elementary Functions
   (i) Constant functions
   (ii) Power functions (**Need** *Gamma function* $\Gamma(x)$)
   (iii) Exponential functions
   (iv) Trigonometric functions ($\sin t, \cos t$ & $\sinh t, \cosh t$)
   (v) Piecewise Continuous Functions (*unit step functions*)

Summary: A short table of Laplace transforms

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s>0)$
t^n $(n \geq 0)$	$\frac{n!}{s^{n+1}}$	$(s>0)$
t^a $(a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$(s>0)$
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\cos kt$	$\frac{s}{s^2+k^2}$	$(s>0)$
$\sin kt$	$\frac{k}{s^2+k^2}$	$(s>0)$
$\cosh kt$	$\frac{s}{s^2-k^2}$	$(s>\lvert k\rvert)$
$\sinh kt$	$\frac{k}{s^2-k^2}$	$(s>\lvert k\rvert)$
$u(t-a)$	$\frac{e^{-as}}{s}$	$(s>0)$

Transforms of Derivatives vs. Differentiation of Transforms

I. Transforms of Derivatives $\dashrightarrow$(**Solve I.V.P.**)

$$
\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\,F(s) - f(0) \\
&= s\,F(s) \qquad (\text{if } f(0) = 0) \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0) \\
&= s^n\,F(s) \qquad (\text{if } f(0) = \cdots = f^{(n-2)}(0) = f^{(n-1)}(0) = 0)
\end{aligned}
$$

• Application: (i) *Linear system*; (ii) *Additional transform techniques*: eg. $\mathcal{L}\{te^{at}\}, \mathcal{L}\{t\sin kt\}$ $\dashrightarrow$ *Much easier by using below!!*

II. Differentiation of Transforms

$$
F'(s) = \mathcal{L}\{-tf(t)\} \Longleftrightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}
$$

$$
F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\} \Longleftrightarrow (-1)^n F^{(n)}(s) = \mathcal{L}\{t^n f(t)\}
$$

Transforms of Integrals vs. Integration of Transforms

I. Transforms of Integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{\mathcal{L}\{f(t)\}}{s} = \frac{F(s)}{s}, \qquad \text{for } s > c.$$

Equivalently, then

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau = \int_0^t \mathcal{L}^{-1}\left\{F(s)\right\}\,d\tau.$$

II. Integration of Transforms

$$\int_s^\infty F(\sigma)\,d\sigma = \mathcal{L}\left\{\frac{f(t)}{t}\right\}, \qquad \text{for } s > c.$$

Equivalently, then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}.$$

I. **Partial Fraction Decomposition:** Rule 1 and Rule 2

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+k^2)^2}\right\}=\frac{1}{2k}t\sin kt.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{2k^3}(\sin kt - kt\cos kt).$$

II. **Translation**

(i) Translation of the s-Axis

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \Leftrightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

(ii) Translation of the t-Axis

$$e^{-as}F(s) = \mathcal{L}\{u(t-a)f(t-a)\} \Leftrightarrow \mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

III. **Convolution**

(i) Definition: $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = (g*f)(t)$

(ii) The Convolution Property Theorem

$$\mathcal{L}\{f(t)*g(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\} \Leftrightarrow \mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f(t)*g(t).$$

Stay safe!

Good Luck for all Finals!!