Solutions homework 9.

(1) Let $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$. Prove that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$. Does (f_n) converge uniformly to f?
Solution: There are 3 cases: $|x| < 1$, $|x| = 1$, and $|x| > 1$. In case $|x| < 1$, then $x^{2n} \to 0$ as $n \to \infty$, which implies that $f_n(x) \to 0$ as $n \to \infty$ for $|x| < 1$. In case $|x| = 1$, then $x^{2n} = 1$ for all n, so $f_n(x) \to \frac{1}{2}$ when $|x| = 1$. In case $|x| > 1$, then divide the numerator and denominator by x^{2n} to see that $f_n(x) \to 1$ when $|x| > 1$. This shows that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x, but f is not continuous at the points $x = \pm 1$, which implies that the convergence is not uniform (as each f_n is continuous and uniform limits of continuous functions are continuous).

(2) Define $f_n : [0,1] \to [0,1]$ by $f_n(x) = x^n(1-x)$. Prove that f_n converges uniformly to 0.
Solution: Note first that from $x^n \geq x^{n+1}$ on $[0,1]$ if follows that $f_{n+1}(x) \leq f_n(x)$. For $0 \leq x < 1$ we have that $x^n \to 0$, so also $f_n(x) \to 0$ for $0 \leq x < 1$. For $x = 1$ we have $f_n(1) = 0$ for all n, so $\lim f_n(x) = 0$ for all $x \in [0,1]$. From Dini's theorem it follows that f_n converges uniformly to 0, since the limit function is continuous, (f_n) is monotone and $[0,1]$ is compact.

(3) Prove that

$$f_n(x) = \frac{nx + \sin(nx^2)}{n}$$

converges uniformly to f on $[0,1]$, where $f(x) = x$.
Solution: $|f_n(x) - f(x)| = |\frac{\sin(nx^2)}{n}| \leq \frac{1}{n}$ for all $x \in [0,1]$, i.e., $d_\infty(f_n, f) \leq \frac{1}{n} \to 0$.

(4) Let $f_n(x) = x^n e^{-nx}$. Prove that $\sum f_n$ converges uniformly on $[0, \infty]$(Hint: Use the Weierstrass M-test).
Solution: Computing the derivative, we find that f_n has a maximum at $x = 1$. Hence $f_n(x)| \leq f_n(1) = (\frac{1}{e})^n$. Since $0 < \frac{1}{e} < 1$ the series $\sum(\frac{1}{e})^n$ converges, thus the series $\sum f_n$ converges uniformly by the Weierstrass M-test.

(5) Prove that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ converges uniformly on $[0,2]$.
Solution: $nx^2 \leq 4n$ on $[0,2]$ and $n^3 + x^3 \geq n^3$ on $[0,2]$. Hence $0 \leq \frac{nx^2}{n^3+x^3} \leq \frac{4n}{n^3} = \frac{4}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, it follows by the Weierstrass M-test that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ converges uniformly on $[0,2]$.