Solutions homework 9.

Problem 1 Solution: By Theorem 38.11 there exist for $\epsilon > 0$ open sets $O \supset A$ and $G \supset [0,1] \setminus A$ such that $\mu(O \cap G) < \epsilon$. Let $F = [0,1] \setminus G$. Then $F = [0,1] \cap G^c$ is closed and $F \subset A$. Also $O \setminus F = O \setminus ([0,1] \setminus G) = O \cap ([0,1] \setminus G)^c = O \cap ([0,1] \cap G^c)^c \subset O \cap G$, which imp[lies that $\mu(O \setminus F) < \epsilon$.

Problem 2 Solution: Since $\mu(A_1) = 1$ we have $\mu([0,1] \setminus A_1) = 0$. Now $A_2 \setminus A_1 \subset [0,1] \setminus A_1$, so $\mu(A_2 \setminus A_1) = 0$. Now $A_2 = (A_1 \cap A_2) \cup A_2 \setminus A_1$ is a disjoint union of measurable sets, so $\mu(A_2) = \mu(A_1 \cap A_2) + \mu(A_2 \setminus A_1) = \mu(A_1 \cap A_2)$.

Problem 3 = Problem 38-10. Solution: If $A \subset G$ with G open then $a + A \subset a + G$. Now $a + G$ open and $m(a + G) = m(G)$. This implies that $\mu^*(a + A) \leq \mu^*(A)$. Similarly $\mu^*(A) \leq \mu^*(a + A)$, so $\mu^*(A) = \mu^*(a + A)$. Remains to show that $a + A$ is measurable. Let $\epsilon > 0$. Then there exists an elementary set B such that $\mu^*(A \triangle B) < \epsilon$. Then $a + B$ is an elementary set and $a + A \triangle B = (a + A) \triangle (a + B)$. Hence $\mu^*(a + A \triangle a + B) < \epsilon$. Hence $a + A$ is measurable.

Problem 4 Solution: Let $A = \{x\}$. The $A \subset (x - \epsilon, x + \epsilon)$ implies that $\mu^*(A) \leq 2\epsilon$ for all $\epsilon > 0$, so $\mu^*(A) = 0$. This shows A is measurable and has measure zero. Now $\mathbb{Q} \cap [0,1]$ can be written as a disjoint union $\cup_n A_n$, where $A_n = \{r_n\}$. Hence $\mu(\mathbb{Q} \cap [0,1]) = \sum_n \mu(A_n) = 0$.

Problem 5 Solution: By problem 4 $\mathbb{Q} \cap [0,1]$ is measurable, so also $[0,1] \setminus \mathbb{Q}$ is measurable. Since $\mu(\mathbb{Q} \cap [0,1]) = 0$ it follows that $\mu([0,1] \setminus Q) = 1 - 0 = 1$.