Solutions homework 9.

(1) **Problem 12-3** First proof: Let $\{b_k\}$ be a sequence in $\cap_k I_k$ which converges to b. Then $\{b_k\}$is a sequence in each I_k, so $b \in I_k$ for all k as I_k is a closed set. Hence $b \in \cap_k I_k$. This shows that $\cap_k I_k$ is closed. Note this proof does not use that the I_k's are intervals or that the collection of intervals is countable.
Second proof: Let $J_k = \cap_{l=1}^k I_k$. Then by hw 9-9 J_k is a closed interval. Now the sequence J_k is a sequence of nested closed intervals, so by the nested interval theorem the intersection $\cap_k J_k$ is a closed interval (possibly consisting of one point). The conclusion follows now, since $\cap_k J_k = \cap_k I_k$.

(2) **Problem 12-4** No, take e.g. $I_k = [\frac{1}{n}, 2 - \frac{1}{n}]$. Then $\cup_n I_n = (0, 2)$.

(3) **Problem 12-6: (1)** Since the sequence converges to e, the upper and lower limit equal e.
(3) Note $-\frac{1}{k} \leq \frac{\sin k}{k} \leq \frac{1}{k}$ impies that $\lim_{k\to\infty} \frac{\sin k}{k} = 0$. On the other hand from calculus we know $\lim_{h\to 0} \frac{\sin h}{h} = 1$ (this limit is used to prove that the derivative of the sine function equals the cosine function and is derived by means of the inequalities $\cos x \leq \frac{\sin x}{x} \leq 1$ for $0 < x < \frac{\pi}{2}$). Putting $h = \frac{1}{k}$ we see that $\lim_{k\to\infty} k \sin \frac{1}{k} = 1$. Therefore the sum has limit 1 and again the upper and lower limit equal the limit, which is 1.

(4) **Problem 12-7** This is false. Take e.g. $a_k = -1$ for all k, so that $a = -1$, and take $b_k = 1 + (-1)^k$. Then $\{b_k\} = (0, 2, 0, 2, \cdots)$ and thus $b^* = \limsup b_k = 2$. Hence $a.b^* = -2$. On the other hand $\{a_k b_k\} = (0, -2, 0, -2, \cdots)$ which shows $\limsup a_k b_k = 0 \neq -2$.

(5) **Problem 12-12** We will show that if $a_k \geq 0$ and $b^* = \limsup b_k \in \mathbb{R}$, then $\limsup a_k b_k = ab^*$. Let $\epsilon > 0$. Then there exists k_0 such that $b_k < b^* + \epsilon$ for all $k \geq k_0$. Hence $a_k b_k \leq a_k(b^* + \epsilon)$ for all $k \geq k_0$. This implies that $\limsup a_k b_k \leq \limsup a_k(b^* + \epsilon) = \lim_{k\to\infty} a_k(b^* + \epsilon) = a(b^* + \epsilon)$. As this holds for all $\epsilon > 0$ we get $\limsup a_k b_k \leq ab^*$. To get the reverse inequality we use that $b_k > b^* - \epsilon$ for infinitely many k. Thus also $a_k b_k \geq a_k(b^* - \epsilon)$ for infinitely many k. If $b^* - \epsilon > 0$, we have $a_k(b^* - \epsilon) > (a - \epsilon)(b^* - \epsilon)$ for infinitely many k, so that $\limsup a_k b_k \geq (a - \epsilon)(b^* - \epsilon)$ for all $\epsilon > 0$, which proves in that case $\limsup a_k b_k \geq ab^*$. If $b^* - \epsilon < 0$, then $a_k(b^* - \epsilon) \geq (a + \epsilon)(b^* - \epsilon)$ for infinitely k and this implies now that $\limsup a_k b_k \geq (a + \epsilon)(b^* - \epsilon)$ for all $\epsilon > 0$, which proves in this case $\limsup a_k b_k \geq ab^*$.