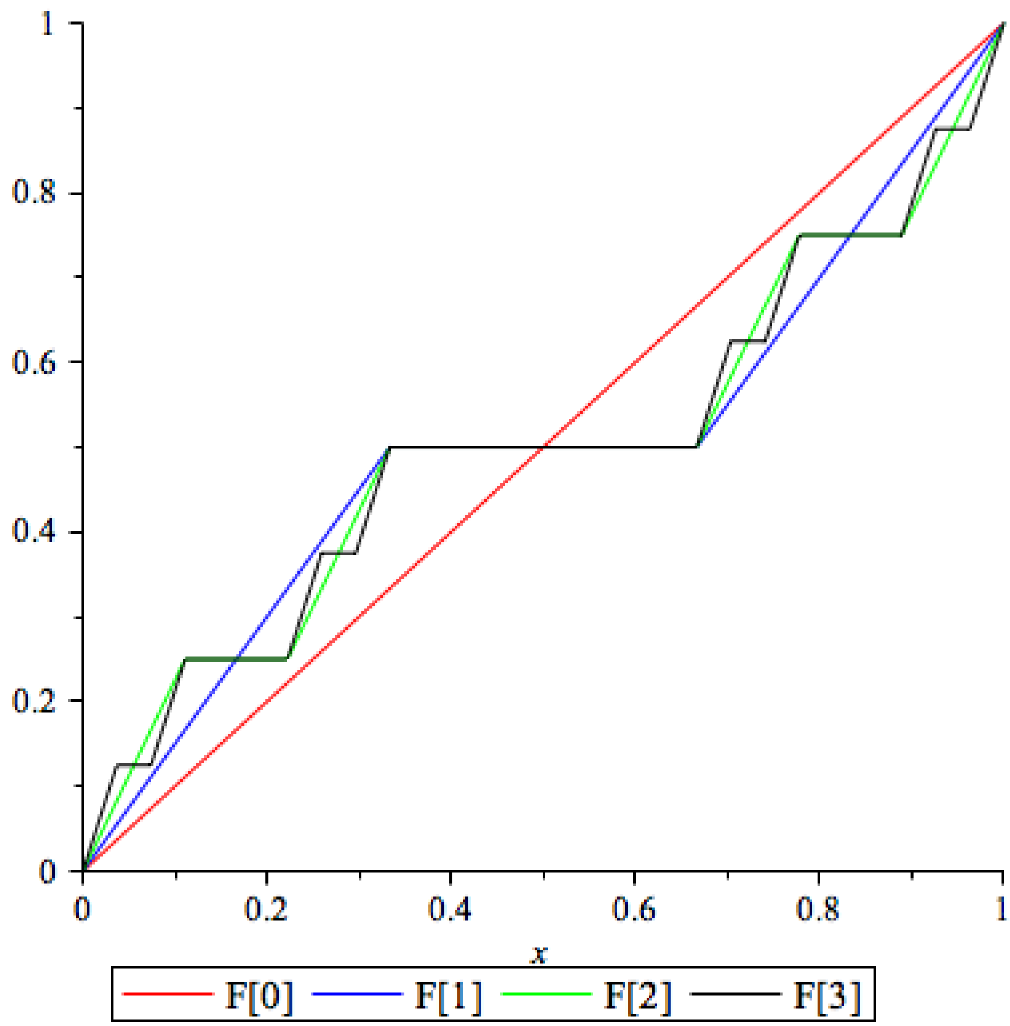
Solutions for HW 9

Exercise 1.6.38: Solution:

(i) Let $x_0 \in \mathbb{R}$. Then $|f(x)| \leq |f(x) - f(x_0)| + |f(x_0)| \leq \|f\|_{TV} + |f(x_0)|$ shows that f is bounded.

(ii) Write $f(x) = f(-M) + \|f\|_{PV[-M,x]} - \|f\|_{NV[-M,x]}$ and take the limit $M \to \infty$. It follows that $\lim_{M\to\infty} f(-M)$ exists. The proof for $+\infty$ is similar.

Exercise 1.6.48: Solution:



(i)

(ii) Clearly F_0 is continuous and monotone with $F_0(0) = 0$ and $F_0(1) = 1$. Assume now that we know this is true for F_{n-1}. Then it is clear that also F_n is continuous (check that this is true for $x = \frac{1}{3}$ and $x = \frac{2}{3}$ separately), as it continuous on each of the three sub-intervals. To prove that F_n is increasing, note that $F_n(x) \leq \frac{1}{2}$ on $[0, \frac{1}{3}]$ and $\frac{1}{2} \leq F_n(x)$ on $[\frac{2}{3}, 1]$ and use that F_{n-1} is increasing.

(iii) Again by induction. $|F_1(x) - F_0(x)| = |\frac{3}{2}x - x| \leq \frac{1}{2}$ on $[0, \frac{1}{3}]$. On $[\frac{1}{3}, \frac{2}{3}]$ we have $|F_1(x) - F_0(x)| = |\frac{1}{2} - x| \leq \frac{1}{2}$, and on $[\frac{2}{3}, 1]$ we have $|F_1(x) - F_0(x)| = |\frac{1}{2}x - \frac{1}{2}| \leq \frac{1}{2}$. hence the inequality holds for $n = 0$. Assume now it holds for n. Then for $x \in [0, \frac{1}{3}]$

we have that

$$|F_{n+1}(x) - F_n(x)| = |\frac{1}{2}F_n(3x) - \frac{1}{2}F_{n-1}(3x)| \leq \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}.$$

Using this estimate we see that for $m > n$ that

$$|F_m(x) - F_n(x)| \leq |F_m(x) - F_{m-1}(x)| + \cdots + |F_{n+1}(x) - F_n(x)| \leq \frac{1}{2^{n-1}}.$$

Hence the sequence $\{F_n\}$ is uniformly Cauchy and converges thus uniformly to a continuous function F. As all F_n are increasing the same holds for F. Moreover $F(0) = 0$ and $F(1) = 1$.

(iv) It is clear that $F_1(x) = \frac{1}{2}$ on the first middle one-third interval. The same holds for all F_n for all $n \geq 1$. Now $F_2(x) = \frac{1}{4}$ 0n $[\frac{1}{9}, \frac{2}{9}]$ and $F_2(x) = \frac{3}{4}$ 0n $[\frac{7}{9}, \frac{8}{9}]$. By induction we can then see that $F_n(x) = \frac{k}{2^n}$ if we number the removed intervals from left to right by $k = 1, \cdots, 2^n - 1$. Since the measure of the union of the removed intervals equals one, we see that $F^{'}(x) = 0$ a.e.

(v) By continuity of F it suffices to show this for $x_N = \sum_{n=1}^{N} a_n 3^{-n}$ with $N \geq 1$. This follows easily if we observe that $F(x_N) = F_k(x_N)$ for all $k \geq N$ and by induction we can show that $F_N(x_N) = F_{N-1}(x_{N-1}) + \frac{a_N}{2} 2^{-N}$.

(vi) This follows from the previous part, by encoding the left endpoints of those intervals in base 3, using 0's and 2's.

(vii) Let $x \in C$. Then there exist unique intervals $I_n = [x_{n,y_n}]$ as in the previous such that $x_n \to x$ and $y_n \to x$. Assume F is differentiable at x and assume first that $x \neq x_n$ and $x \neq y_n$ for all n. Then

$$\frac{F(x) - F(x_n)}{x - x_n} \to F^{'}(x)$$

and

$$\frac{F(x) - F(y_n)}{x - y_n} \to F^{'}(x).$$

This implies that also

$$\frac{F(y_n) - F(x_n)}{y_n - x_n} \to F^{'}(x).$$

However by the previous item

$$\frac{F(y_n) - F(x_n)}{y_n - x_n} = \frac{3^n}{2^n},$$

which leads to a contradiction. If x is one of the endpoints we only use the other endpoint, but the rest of the argument is the same.

Problem 1: Solution: Let $h > 0$. Then $\frac{F(h)-F(0)}{h} = \sin 1/h$. Hence $\overline{D^+}F(0) = 1$, $\underline{D^+}F(0) = -1$, $\overline{D^-}F(0) = 1$, and $\underline{D^-}F(0) = -1$.

Problem 2: Solution: Let $h > 0$. Then $F(c+h) \geq F(c)$ for all h small enough. Thus for all $\delta > 0$ we have $\inf_{0<h<\delta}(F(c+h) - F(c))/h \geq 0$. Thus $0 \leq \underline{D^+}F(c)$. The other inequality follows similarly.