Solutions homework 8.

Page 188 Problem 6. $\int_1^n \frac{1}{\sqrt{x}}\,dx = 2\sqrt{n} - 2 \to \infty$ as $n \to \infty$. The divergence of the series follows now from

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{n}} \geq \int_1^n \frac{1}{\sqrt{x}}\,dx.$$

A simpler proof follows from $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ and the fact that $\sum \frac{1}{n}$ diverges.

Page 188 Problem 7. From $\ln x \leq x - 1$ it follows that $\ln(1+n) \leq (1+n) - 1 = n$. Hence $\frac{1}{\ln(1+n)} \geq \frac{1}{n}$, which shows that the series $\sum \frac{1}{\ln(1+n)}$ diverges.

Page 188 Problem 10. Multiply the first n quotients to get

$$\frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdots \frac{a_{n+1}}{a_n} \leq \frac{b_2}{b_1} \cdot \frac{b_3}{b_2} \cdots \frac{b_{n+1}}{b_n},$$

which, by canceling common terms, is the same as $\frac{a_{n+1}}{a_1} \leq \frac{b_{n+1}}{b_1}$. The convergence of $\sum a_n$ follows now from the comparison test.

Page 188 Problem 11.

a. If $\frac{a_{n+1}}{a_n} \geq 1$ ultimately, then take now $b_n = 1$. Then $\sum b_n$ diverges and $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$ ultimately implies, via the contrapositive of Problem 10, that $\sum a_n$ diverges.

b. If $\limsup \frac{a_{n+1}}{a_n} < 1$, then there exists $0 < r < 1$ such that $\frac{a_{n+1}}{a_n} < r$ ultimately. Take now $b_n = r^n$ and apply Problem 10 to get convergence.

c. Let $0 < c < 1$ and define $a_{2n-1} = c^n$ and $a_{2n} = 2c^n$. Then $\frac{a_{2n}}{a_{2n-1}} = 2$ and $\frac{a_{2n+1}}{a_{2n}} = \frac{c}{2}$. It follows that $\limsup \frac{a_{n+1}}{a_n} = 2 > 1$, but the series converges (its sum is equal to $\frac{3}{1-c}$).

d. Define $a_{2n-1} = \frac{1}{n^3}$ and $a_{2n} = \frac{1}{n^2}$. Then $\frac{a_{2n}}{a_{2n-1}} = n$, but the series converges (it is the sum of two convergent series)

Page 188 Problem 13. $\frac{a_{n+1}}{a_n} = \frac{c}{n+1}$, so $\limsup \frac{a_{n+1}}{a_n} = 0$, so the series converges by the Ratio test.

Page 193 Problem 7. Assume $|b_n| \leq M$. Then $|a_n b_n| \leq M|a_n|$. Now $\sum M|a_n|$ converges, so by the comparison test $\sum |a_n b_n|$ converges.

Page 193 Problem 8.

a. From Problem 7 above we know that $\sum \frac{1}{\ln(1+n)}$ diverges. On the other hand, by the alternating series test, $\sum \frac{(-1)^n}{\ln(1+n)}$ converges, so it is conditionally convergent.

b. Since $\limsup \sqrt[n]{\frac{1}{n^n}} = 0$ it follows that $\sum \frac{1}{n^n}$ converges. Now $|(\sin n)^n| \leq 1$, so that the given series converges absolutely by Problem 7.