Solutions for HW 8

Exercise 1.5.9: Solution: If f_n converges to f in L^1 norm, then f_n converges to f in measure by Markov's inequality (no domination needed for this implication). Now assume f_n converges to f in measure and that (f_n) is dominated by some integrable g. Assume that f_n does not converges to f in L^1 norm. Then there exists $\epsilon > 0$ and a subsequence f_{n_k} such that $\|f - f_{n_k}\|_1 \geq \epsilon$ for all k. Now this subsequence converges still in measure to f, so by what was shown in class it contains a further subsequence $f_{n_{k_l}}$ which converges pointwise to f a.e. Now by the Dominated Convergence Theorem (actually a consequence of it) this subsequence converges in norm to f, which contradicts $\|f - f_{n_k}\|_1 \geq \epsilon$ for all k.

Exercise 1.5.18: Solution: If f_n converges almost uniformly to f, then f_n converges pointwise a.e. to f as shown in class (no domination needed for this implication). Assume now that f_n converges to f a.e and that (f_n) is dominated by some integrable g. Then one can check the proof of Egorov's theorem as given in class to see that the exact same proof still works. One uses the fact that

$$E_{N,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k} \text{ for some } n \geq N\} \subset \{x \in X : 2g(x) \geq \frac{1}{k}\},$$

since $|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq 2g(x)$ and that the set on the right has finite measure. This implies the $\lim_{N\to\infty} m(E_{N,k}) = 0$ and the rest is identical to the proof of Egorov's theorem.

Exercise 1.6.1: Solution: Define $F_n(x) = n(F(x + \frac{1}{n}) - F(x))$. Then F_n is continuous and $F_n(x) \to F'(x)$ at every point $x \in (a, b)$, where F is differentiable. This implies the measurability claims of F'. It is elementary to show that if F is differentiable, then F is continuous. For a counter example for the continuity claim take $[a, b] = [-1, 1]$ and $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Then F' exists a.e. (everywhere except at $x = 0$) and F is discontinuous at 0.

Problem 1: Solution: For $x \neq 0$ we get from the product and chain rule that $F'(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2)$. For $x = 0$ we use the definition to see that

$$\lim_{h\to 0} \frac{F(h) - F(0)}{h} = \lim_{h\to 0} h \sin(1/h^2) = 0.$$

Hence F is everywhere differentiable. To see that F' is not integrable, we observe first that

$$\int_{-1}^{1} |2x \sin(1/x^2)|\, dx \leq \int_{-1}^{1} 2|x|\, dx = 2 < \infty.$$

Hence it suffices to show $\int_0^1 \frac{2}{x} |\cos(1/x^2)|\, dx = \infty$. Changing the variable to $y = 1/x^2$ this is the same as showing that $\int_1^\infty \frac{1}{y} |\cos(y)|\, dy = \infty$. To see this note

$$\int_1^\infty \frac{1}{y} |\cos(y)|\, dy \geq \sum_{n=1}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\cos(y)|\, dy = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$