Solutions Homework 8.

(1) Prove that if $z = x + iy$ and $f(z) = \sqrt{(|xy|)}$, then the real part and imaginary part of f satisfy the Cauchy-Riemann equations at $z = 0$, but f is not differentiable at $z = 0$.
Solution: $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$. Hence $\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0$ for all (x,y). To find $\frac{\partial u}{\partial x}(0,0)$ we use the definition:

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0, h\in\mathbb{R}} \frac{u(h,0)-u(0,0)}{h} = 0.$$

Similarly $\frac{\partial u}{\partial y}(0,0) = 0$. Hence the Cauchy-Riemann equations hold at $z = 0$. On the other hand

$$\frac{f(t(1+i)) - f(0)}{t(1+i)} = \frac{|t|}{t(1+i)}$$

does not have a limit as $t \to 0$ in $\mathbb{R}$. Hence $f'(0)$ does not exist.

(2) Let $G \subset \mathbb{C}$ be an open and connected set and let $f : G \to \mathbb{C}$ be a holomorphic function such that $f'(z) = 0$ for all $z \in G$. Prove that f is constant on G. (Hint: let $S = \{z \in G : f(z) = f(z_0)\}$ for some fixed $z_0 \in G$ and show that S is open and closed. You can use from undergraduate analysis that if a real valued differentiable function has zero derivative on an interval, then that function is constant on the interval.)
Solution: Let $z_0 \in G$ and put $S = \{z \in G : f(z) = f(z_0)\}$. Then the continuity of f implies that S is closed. Let now $a \in S$. Then there exists $\epsilon > 0$ such that $B(a;\epsilon) \subset G$. Let $z \in B(a;\epsilon)$ and put $g(t) = f(tz + (1-t)a)$ for $0 \le t \le 1$. Then by the chain rule $g'(t) = f'(tz + (1-t)a)(z-a) = 0$ for $0 < t < 1$, so g is constant on $0 \le t \le 1$. Hence $f(z) = g(0) = g(1) = f(z_0)$, and thus $BD(a;\epsilon) \subset S$. It follows that S is nonempty open and closed subset of G, thus $S = G$.

(3) Let f be holomorphic on the unit disk $B(0;1)$.
   a. Prove that if $\operatorname{Re} f$ is constant on $B(0;1)$, then f is constant.
   Solution: $u(x,y) = c$ implies $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ on $x^2 + y^2 < 1$. From the Cauchy-Riemann equations we get that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ on $x^2 + y^2 < 1$. This implies that $v(x,y) =$ constant on $x^2 + y^2 < 1$. Hence $f = u + iv$ is constant on $x^2 + y^2 < 1$.
   b. Prove that if e^f is constant on $B(0;1)$, then f is constant.
   Solution: If $e^{f(z)} = c$, then $0 = (e^{f(z)})' = f'(z)e^{f(z)}$. As $e^f \neq 0$, this implies that $f'(z) = 0$ for all $|z| < 1$. This implies that $f = c$ (by the previous problem).

(4) Let $G \subset \mathbb{C}$ be open and let f be holomorphic on G. Let $G^* = \{z : \bar{z} \in G\}$ and define $f^*(z) = \overline{f(\bar{z})}$ for all $z \in G^*$. Prove that f^* is holomorphic on G^* and express $f^*(z)'$ in terms of f'.
Solution: For $z \in G^*$ we have

$$\frac{f^*(z+h) - f^*(z)}{h} = \overline{\left(\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}\right)} \to \overline{f'(\overline{z})}$$

as $h \to 0$. Hence f^* is holomorphic on G^* and $f^*(z)' = \overline{f'(\overline{z})}$

(5) (Quals '04) Let $G \subset \mathbb{C}$ be an open and connected set and let $f : G \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = C$ for all $z \in G$. Prove that f is constant on G.

Solution: Note if $C = 0$, then we are done. Assume $C \neq 0$. Let $f = u + iv$. Then $|f(z)|^2 = u^2 + v^2 = C^2$. Taking partial derivatives w.r.t. x and y we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

on G. Using the Cauchy–Riemann equations we can rewrite the second equation as

$$-2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = 0.$$

Multiplying the first equation by u and the third one by v and adding the the two resulting equations we obtain

$$2(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

and thus $\frac{\partial u}{\partial x} = 0$ on G. Similarly one sees that $\frac{\partial v}{\partial x} = 0$ on G and thus $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$ on G. This implies that f is constant on G.