Solutions homework 7.

Page 182 Problem 1. The answer is 2 times the sum of the geometric series with $c = \frac{1}{3}$, so the answer is $2 \cdot \frac{1}{1-\frac{1}{3}} = 3$.

Page 182 Problem 4. For $n \geq 1$ we have $s_{2n+2} = s_{2n} + a_{2n+1} + a_{2n+2} \geq s_{2n}$ (since $a_{2n+1} + a_{2n+2} \geq 0$), so (s_{2n}) is an increasing sequence. Similarly $s_{2n+1} = s_{2n-1} + a_{2n} + a_{2n+1} \leq s_{2n-1}$, so (s_{2n-1}) is a decreasing sequence. Moreover $s_{2n+1} = s_{2n} + a_{2n+1} \geq s_{2n}$, From this it follows that (s_{2n}) is bounded above by s_1 and that (s_{2n+1}) is bounded below by s_2. Hence both limits $\lim s_{2n}$ and $\lim s_{2n+1}$ exist. Now $s_{2n+1} = s_{2n} + a_{2n+1}$ and $a_{n+1} \to 0$ show that these two limits are equal. Hence $\lim s_n$ exists and the series converges.

Page 184 Problem 1. If (s_n) is bounded, then by Bolzano–Weierstrass this sequence gas a convergent subsequence (s_{n_k}). Using these $n_k's$ for the grouping, we get as in the proof of Theorem 10.2.3 that $(t_k) = (s_{n_k})$ converges.

Page 187 Problem 2. Let $s_n = a_1 + \cdots + a_n$ and $t_n = b_1 + \cdots + b_n$. Then $\sum a_n$ converges if and only if (s_n) is bounded, and $\sum b_n$ converges if and only if (t_n) is bounded, since both series are positive term series. Hence If $\sum b_n$ converges, then (t_n) is bounded, which implies that (s_n) is bounded and thus $\sum a_n$ converges whenever $\sum b_n$ converges.

Extra Problem 1. Let $\epsilon > 0$ and take $\delta > 0$ as in the hint. Then take $\frac{1}{n} < \delta$ and take $\sigma = \{0 < \frac{1}{n} < \cdots < \frac{k}{n} < \cdots < \frac{n}{n} = 1\}$. Then $s(\sigma) \leq \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) \leq \mathcal{S}(\sigma)$ and $\mathcal{S}(\sigma) - s(\sigma) < \epsilon$. Hence $|\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) - \int_0^1 f| < \epsilon$.

Extra Problem 2.

a. Take $f(x) = x^2$. Then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k^2,$$

so

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \int_0^1 x^2\,dx = \frac{1}{3}.$$

b. Take $f(x) = \frac{1}{1+x^2}$. Then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \frac{n}{n^2+k^2},$$

so

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \arctan x|_0^1 = \frac{\pi}{4}.$$