Solutions homework 7.

Problem 1 Solution: Let $\epsilon > 0$. Then by the proof of Theorem 2.1 there exists $\delta > 0$ such that for any partition $\mathcal{P}$ of $[0,1]$ with mesh $|\mathcal{P}| < \delta$ we have

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon.$$

Now take $\mathcal{P} = \{x_0 = 0, x_1 = \frac{1}{n}, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1\}$, where $n \geq N$ and $\frac{1}{N} < \delta$. Then for all $n \geq N$ we have

$$\mathcal{L}(\mathcal{P}, f) \leq \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \mathcal{U}(\mathcal{P}, f).$$

This implies that also

$$\left| \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_a^b f(x)\, dx \right| < \epsilon$$

for all $n \geq N$, which completes the proof.

Problem 2 Solution: To apply Problem 1 we need that $\frac{1}{n} f(\frac{k}{n}) = \frac{k}{n^2+k^2}$ for all k and n. Therefore take $f(x) = \frac{x}{1+x^2}$. By problem 1 the limit equals

$$\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2} \ln 2.$$

Problem 3 Solution: By the chain rule

$$\frac{d}{dx} \int_a^{h(x)} f(t)\, dt = f(h(x))h'(x)$$

and similarly

$$\frac{d}{dx} \int_{g(x)}^{b} f(t)\, dt = -f(g(x))g'(x).$$

Now

$$\int_{h(x)}^{g(x)} f(t)\, dt = \int_a^b f(t)\, dt - \int_a^{h(x)} f(t)\, dt - \int_{g(x)}^{b} f(t)\, dt,$$

which shows that

$$\frac{d}{dx} \int_{h(x)}^{g(x)} f(t)\, dt = f(g(x))g'(x) - f(h(x))h'(x).$$

Problem 4 Solution: By the notes $\int_a^x f(t)\, dt$ is continuous, so since compostion of continuous functions is continuous it follows that g is continuous. To find $g'(x)$ use the previous problem to get $g'(x) = f(x+a) - f(x-a)$.

Problem 5 Solution: Let $\epsilon > 0$. Then there exist $b < 1$ so that $\int_b^1 |f(x^n)|\, dx + |f(0)|(1-b) < \frac{\epsilon}{2}$. Now the sequence $f_n(x) = f(x^n)$ converges uniformly to $f(0)$ on $[0, b]$, so there exists N such that

$$\left| \int_0^b f(x^n)\, dx - f(0)b \right| < \frac{\epsilon}{2}$$

for all $n \geq N$. Now it follows that

$$\left| \int_0^1 f(x^n)\, dx - f(0) \right| \leq \left| \int_0^b f(x^n)\, dx - f(0)b \right| + \int_b^1 |f(x^n)|\, dx + |f(0)|(1-b) < \epsilon$$

for all $n \geq N$.