Solutions homework 7.

(1) **Problem 11-1.** (3) Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Let $\epsilon > 0$. Then there exists $n_0, n_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ for all $n \geq n_0$ and $|b_n - b| < \frac{\epsilon}{2}$ for all $n \geq n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq n_2$.

(4) Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Let $\epsilon > 0$. Then $\{a_n\}$ convergent, implies that it is bounded, so there exists $M$ such that $|a_n| \leq M$ for all $n \geq 1$. Now there exists $n_0, n_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2(|b|+1)}$ for all $n \geq n_0$ and $|b_n - b| < \frac{\epsilon}{2M}$ for all $n \geq n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then

$$|a_n b_n - ab| \leq |a_n b_n - a_n b + a_n b - ab| \leq |a_n||b_n - b| + |a_n - a||b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq n_2$.

(2) **Problem 11-2** First example: take $a_n = n$ and $b_n = -n$. Second example: take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

(3) **Problem 11-3.** From $\lim_{n\to\infty}(a_n + b_n) = c$ it follows that $\lim_{n\to\infty}(a_n + b_n)^2 = c^2$ by 11-1 (4). Similarly $\lim_{n\to\infty}(a_n - b_n) = d$ implies that $\lim_{n\to\infty}(a_n - b_n)^2 = d^2$. Now the result follows from $a_n b_n = \frac{1}{4}\left((a_n + b_n)^2 - (a_n - b_n)^2\right)$.

(4) **Problem 11-4** If $\{a_k\}$ converges, then also $\{-a_k\}$ converges. Now the result follows from $b_k = (a_k + b_k) + (-a_k)$ and 11-1 (3).

(5) **Problem 11-5** No, take e.g. $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$. Then $a_k + b_k = 0$ and $a_k b_k = -1$ for all $k \geq 1$.