Solutions for HW 7

Exercise 1.4.46: Solution: By Fatou's lemma

$$\int |f(x)|\,dx \le \underline{\lim} \int |f_n(x)|\,dx \le \underline{\lim} \int G(x) + g_n(x)\,dx \le \int G(x)\,dx < \infty.$$

Hence f is also integrable. Another application of Fatou's lemma shows

$$\int \underline{\lim}\, g_n(x)\,dx = 0.$$

As in the proof of the DCT we can assume that all functions are real valued. Then $G + g_n + f_n \ge 0$ a.e. and $G + g_n - f_n \ge 0$ a.e. Hence by applying Fatou's lemma twice we get

$$\int G(x) + \underline{\lim}\, g_n(x) + f(x)\,dx \le \underline{\lim} \int G(x) + g_n(x) + f_n(x)\,dx = \int G(x)\,dx + \underline{\lim} \int f_n(x)\,dx$$

and

$$\int G(x) + \underline{\lim}\, g_n(x) - f(x)\,dx \le \underline{\lim} \int G(x) + g_n(x) - f_n(x)\,dx = \int G(x)\,dx - \overline{\lim} \int f_n(x)\,dx.$$

Cancelling the common and zero terms results in

$$\overline{\lim} \int f_n(x)\,dx \le \int f(x)\,dx \le \underline{\lim} \int f_n(x)\,dx,$$

from which the claims follows.

Exercise 1.4.47: First Solution: Observe first

$$0 \le |f - f_n| + (f - f_n) = 2(f - f_n)^+ \le 2f.$$

Now $(f - f_n)^+(x) \to 0$ a.e., so by the DCT $\int (f - f_n)^+(x)\,dx \to 0$ and the conclusion follows.
Second Solution: By the Dominated convergence Theorem $\int \min(f, f_n)(x)\,dx \to \int f(x)\,dx$. The result follows then from the identity $f - \min(f_n, f) = (f - f_n)^+ = \frac{1}{2}(|f - f_n| + f - f_n)$.
Problem 1: Solution: Let $\epsilon > 0$ and let g be a continuous function with compact support, say contained in $B(0, N)$. Then the fact that g is uniformly continuous implies that there exists $0 < \delta < 1$ such that $|g(x + h) - g(x)| < \epsilon/3(m(B(0, N + 1)))$ for all $|h| < \delta$ and all $x \in \mathbb{R}^d$ (note that, if $|x| > N + 1$, then both $g(x) = 0$ and $g(x + h) = 0$). This implies $\|g - g_h\|_1 < \epsilon/3$ for all $|h| < \delta$. Now by the approximation theorem for f integrable there exists g with compact support such that $\|f - g\|_1 < \epsilon/3$. Then also $\|f_h - g_h\|_1 < \epsilon/3$ by translation invariance of the Lebesgue integral. Now let $\delta > 0$ be as above. Then $\|f - f_h\|_1 \le \|f - g\|_1 + \|g - g_h\|_1 + \|g_h - f_h\|_1 < \epsilon/3 + \epsilon/3 + \epsilon/3 < \epsilon$ for all $|h| < \delta$.
Problem 2: Solution: First let $f = \chi_{(a,b)}$. Then $\left|\int_{-\infty}^{\infty} f(x)e^{inx}\,dx\right| = \left|\int_a^b e^{inx}\,dx\right| = \left|\frac{e^{inb} - e^{ina}}{in}\right| \to 0$ as $n \to 0$, since $|e^{inb} - e^{ina}| \le 2$. Let now $\epsilon > 0$ and find a step function ψ such that $\int |f - \psi|\,dx < \epsilon/2$. Then there exists N such that $|\int_{-\infty}^{\infty} \psi(x)e^{inx}\,dx| < \epsilon/2$ for all $n \ge N$. Now $\left|\int_{-\infty}^{\infty} f(x)e^{inx}\,dx\right| \le \int |f - \psi||e^{inx}|\,dx + |\int_{-\infty}^{\infty} \psi(x)e^{inx}\,dx| < \epsilon$ for all $n \ge N$.