Solutions homework 6.

Problem 1 Solution: Since f is piecewise continuous on $[c,1]$, it follows that f is Riemann integrable on $[c,1]$. Moreover if N is the largest n such that $c \leq \frac{1}{n}$, then $\int_c^1 f(x)\,dx = \sum_{n=1}^{N-1} \int_{\frac{1}{n+1}}^{\frac{1}{n}} f(x)\,dx = 0$, since f is zero on $(\frac{1}{n+1}, \frac{1}{n})$. Part **b.** follows now from Lemma 2.4 of the notes.

Problem 2 Solution: Part **a.** Let $0 < \epsilon < \frac{1}{2}|f(c)|$. Then there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in (c-\delta, c+\delta)$. This implies that $|f(x)| \geq \frac{1}{2}|f(c)|$ for all $x \in (c-\delta, c+\delta)$. Hence $\frac{1}{2}|f(c)|\chi_{(x-\delta,c+\delta)} \leq |f(x)|$ on $[a,b]$. Integrating we get $0 < \delta|f(c)| \leq \int_a^b |f(x)|\,dx$. Part **b.** is in one direction the contrapositive of part **a.**. If $f(x) = 0$ for all x, then obviously $\int_a^b |f(x)|\,dx = 0$, which is the other direction. For a counterexample in part **c.** take $f(x) = x$ on $[-1,1]$.

Problem 3 Solution: First proof: Use the Fundamental Theorem of Calculus. Second Proof: From the hypothesis it follows that $\int_c^d f(x)\,dx = \int_a^d f(x)\,dx - \int_a^c f(x)\,dx = 0 - 0 = 0$ for all $a \leq c < d \leq b$. Now we can give a proof by contradiction along the lines of the solution of Problem 2: If $f(x_0) \neq 0$ for some x_0, then by continuity we can find an interval $[c,d]$ around x_0 on which $|f(x)|$ is bounded away from zero and does not change sign. Now an estimate as in the solution of problem (2) **a.** shows that $\int_c^d f(x)\,dx \neq 0$.

Problem 4 Solution: Let $\epsilon > 0$ and $|f(x)| \leq M$. Then there exists $0 < c < 1$ such that $(1-c)M < \frac{\epsilon}{2}$. Then $x^n f(x)$ converges uniformly to zero on $[0,c]$, which implies that there exists N such that $|\int_0^c x^n f(x)\,dx| < \frac{\epsilon}{2}$ for all $n \geq N$ Hence

$$\left|\int_0^1 f(x)\,dx\right| \leq \left|\int_0^c x^n f(x)\,dx\right| + M(1-c) < \epsilon$$

for all $n \geq N$.

Problem 5 Solution: Since $|f(x)| \leq M$ the right hand inequality follows immediately. Since f is continuous there exists $x_0 \in [a,b]$ such that $|f(x_0)| = M > 0$. Then by continuity, similar to the solution of problem (2)**a.** we can find for each $\epsilon > 0$ an interval $[c,d] \subset [a,b]$ with $x_0 \in [c,d]$ such that $|f(x)| \geq M-\epsilon$ on $[c,d]$. This implies that $|f(x)|^n \geq (M-\epsilon)^n \chi_{[c,d]}(x)$ and by integrating over $[a,b]$ we get the left hand inequality. Taking now n-th roots and letting $n \to \infty$ we get that $M - \epsilon \leq \lim \int_a^b |f(x)|^n\,dx \leq M$ for all $\epsilon > 0$. This gives the limit of part **b.**