Solutions homework 6.

(1) **Problem 9-2.** (a) A is not bounded below or above.
(b) $B = \{\eta | \eta^2 \leq 5\}$, so glb $B = -\sqrt{5}$ and lub $B = \sqrt{5}$.
(c) C is not bounded below and lub $C = 0$.
(d) Same as (b)
(e) Same as (a)
(f) $F = \{\xi || \xi| \leq 1\}$, so glb $F = -1$ and lub $F = 1$.
(g) $G = (-1, 0)$, so glb $G = -1$ and lub $G = 0$.

(2) **Problem 9-7.** False, take e.g. $I_n = (0, \frac{1}{n})$. Then $\cap_n I_n = \emptyset$. Note the intersection $\cap_{n=1}^{\infty} I_k = \emptyset$ or a single point in general. To see this replace each I_k by the closed interval J_k with the same endpoints as I_k. Then the J_k's form a nested sequence of closed intervals with $m(J_k) \to 0$, so $\cap_{k=1}^{\infty} J_k$ is a single point. Now $\cap_{k=1}^{\infty} I_k$ is equal to that single point or $\emptyset$. To get an example with a single point take $I_k = (\frac{1}{k}, \frac{1}{k})$.

(3) **Problem 9-8** If $I_1 \supset I_2 \supset \cdots \supset I_n$, then $\cap_{k=1}^{n} I_k = I_n$.

(4) **Problem 9-9** First Solution: Prove by induction. It is obviously true for $n = 1$. Assume it holds for $n - 1$. Then $I = \cap_{k=1}^{n} I_k = \cap_{k=1}^{n-1} I_k \cap I_n$. By the induction hypothesis we know that $A = \cap_{k=1}^{n-1} I_k$ is either $\emptyset$, a singleton, or a closed interval. Now it is easy to see that $A \cap I_n$ is again either $\emptyset$, a singleton, or a closed interval. Second Solution: Let $I_k = [\alpha_k, \beta_k]$. If $\cap_{k=1}^{n} I_k = \emptyset$, then we are done. Otherwise let $x \in I_k$ for all $k = 1, \cdots, n$. Then $\alpha_k \leq x \leq \beta_k$ for all $k = 1. \cdots, n$. Thus $\alpha = \max\{\alpha_k : k = 1, \cdots, n\} \leq x \leq \beta = \min\{\beta_k : k = 1, \cdots, n\}$, so $x \in [\alpha, \beta]$. This shows $\cap_{k=1}^{n} I_k \subset [\alpha, \beta]$. We claim that $\cap_{k=1}^{n} I_k = [\alpha, \beta]$, which will prove the assertion of the problem. Since $\alpha_k \leq \alpha \leq \beta \leq \beta_k$, we have $[\alpha, \beta] \subset I_k$ for all $k = 1. \cdots, n$, so $[\alpha, \beta] \subset \cap_{k=1}^{n} I_k$ and we are done.